Principles of Communications

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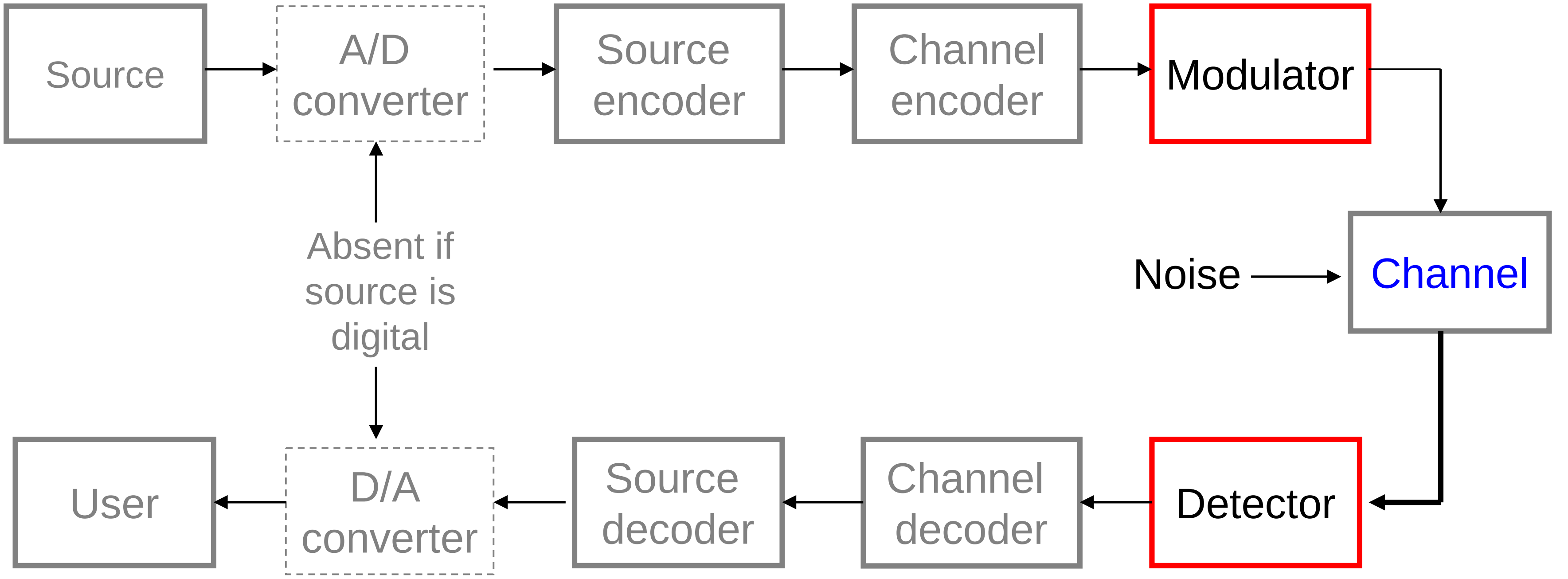
Chapter 8: Digital Modulation Techniques

Selected from Chapter 10.1 – 10.5 of *Fundamentals of Communications Systems*, Pearson Prentice Hall 2005, by Proakis & Salehi

Topics to be Covered



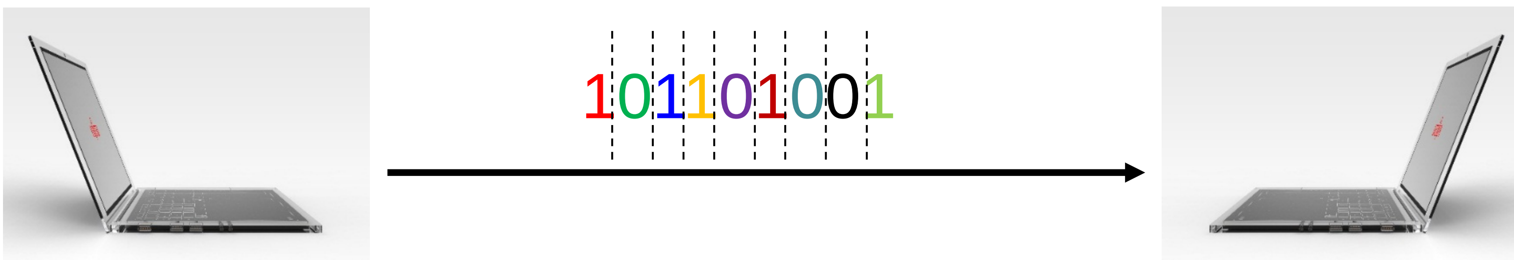
- Binary digital modulation
- M-ary digital modulation
- Comparison study

Digital Modulation

- In digital communications, the modulation process corresponds to switching or keying the amplitude, frequency, or phase of a sinusoidal carrier wave according to incoming digital data
- Three basic digital modulation techniques
  - Amplitude-shift keying (ASK) - special case of AM
  - Frequency-shift keying (FSK) - special case of FM
  - Phase-shift keying (PSK) - special case of PM
- Will use signal space approach in receiver design and performance analysis

Binary Modulation

- In binary signaling, the modulator produces one of two distinct signals in response to one bit of source data at a time.

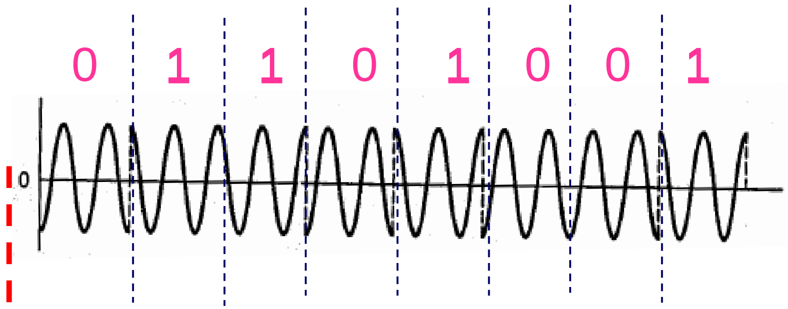


- Binary modulation types

Binary PSK | **Binary FSK** | **Binary ASK**

Binary Phase-Shift Keying (BPSK)

- Modulation



"1" → $s_1(t) = \sqrt{\frac{2E_b}{T_b}} \cos(2\pi f_c t)$

"0" → $s_2(t) = \sqrt{\frac{2E_b}{T_b}} \cos(2\pi f_c t + \pi) = -\sqrt{\frac{2E_b}{T_b}} \cos(2\pi f_c t)$

  - $0 \le t < T_b$, T_b bit duration
  - f_c : carrier frequency, chosen to be n_c/T_b for some fixed integer n_c or $f_c >> 1/T_b$
  - E_b: transmitted signal energy per bit, i.e.

$$\int_0^{T_b} s_1^2(t)dt = \int_0^{T_b} s_2^2(t)dt = E_b$$

  - The pair of signals differ only in a 180-degree phase shift

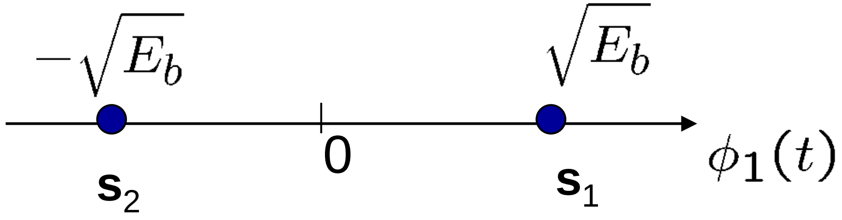
Signal Space Representation for BPSK

- There is one basis function

$$\phi_1(t) = \sqrt{\frac{2}{T_b}}\cos(2\pi f_c t) \quad \text{with} \quad 0 \le t < T_b$$

- Then $s_1(t) = \sqrt{E_b}\phi_1(t)$ and $s_2(t) = -\sqrt{E_b}\phi_1(t)$

- A binary PSK system is characterized by a signal space that is one-dimensional (i.e. $N=1$), and has two message points (i.e. $M=2$)



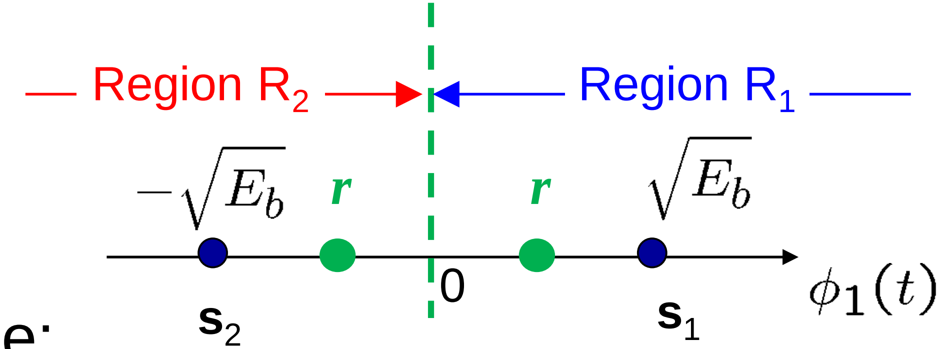
$$d_{12} = 2\sqrt{E_b}$$

Decision Rule of BPSK

- Assume that the two signals are equally likely, i.e.

$$P(s_1) = P(s_2) = 0.5$$

- Then the optimum decision boundary is the midpoint of the line joining these two message points



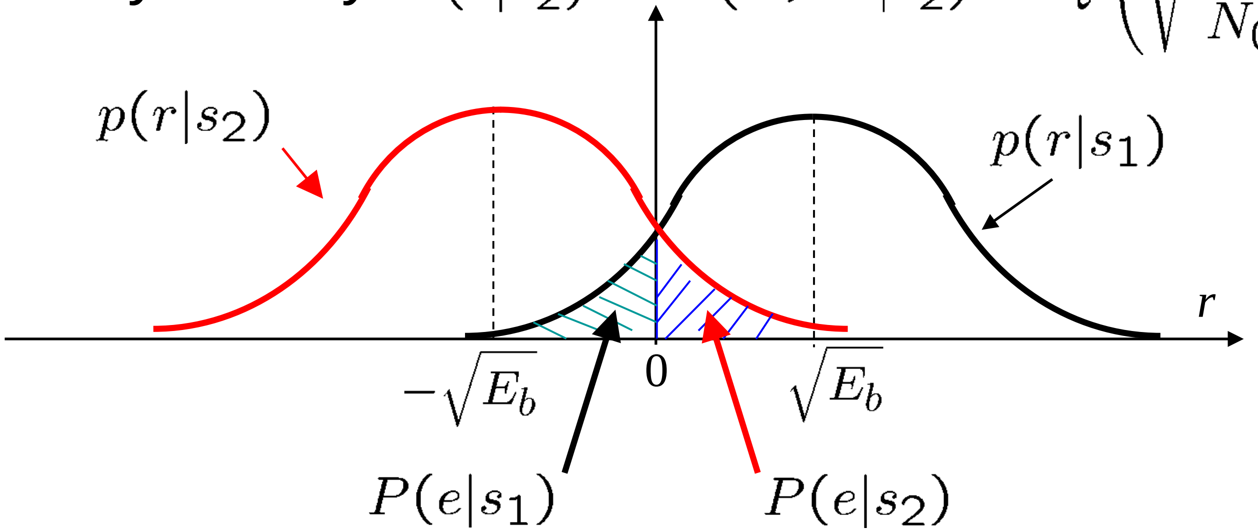
- Decision rule:
  - Guess signal $s_1(t)$ (or binary 1) was transmitted if the received signal point r falls in region R_1 $(r > 0)$
  - Guess signal $s_2(t)$ (or binary 0) was transmitted otherwise $(r \leq 0)$

Probability of Error for BPSK

- The conditional probability of the receiver deciding in favor of $s_2(t)$ given that $s_1(t)$ is transmitted is

$$P(e|s_1) = P(r < 0|s_1)$$
$$= \int_{-\infty}^{0} \frac{1}{\sqrt{\pi N_0}} \exp\left\{-\frac{(r-\sqrt{E_b})^2}{N_0}\right\} dr = Q\left(\sqrt{\frac{2E_b}{N_0}}\right)$$

- Due to symmetry $P(e|s_2) = P(r > 0|s_2) = Q\left(\sqrt{\frac{2E_b}{N_0}}\right)$



P_e for BPSK (cont'd)

- Since the signals $s_1(t)$ and $s_2(t)$ are equally likely to be transmitted, the average probability of error is

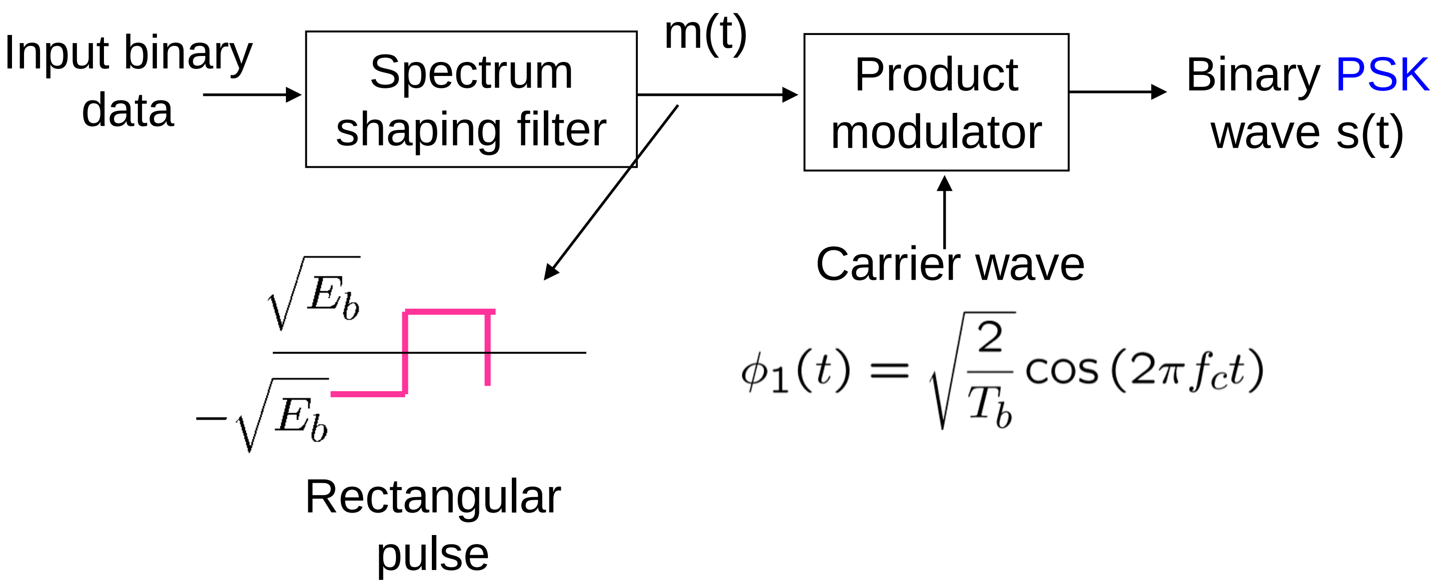
$$P_e = 0.5P(e|\mathbf{s}_1) + 0.5P(e|\mathbf{s}_2) = Q\left(\sqrt{\frac{2E_b}{N_0}}\right)$$

⇓

P_e depends on ratio $\frac{E_b}{N_0}$

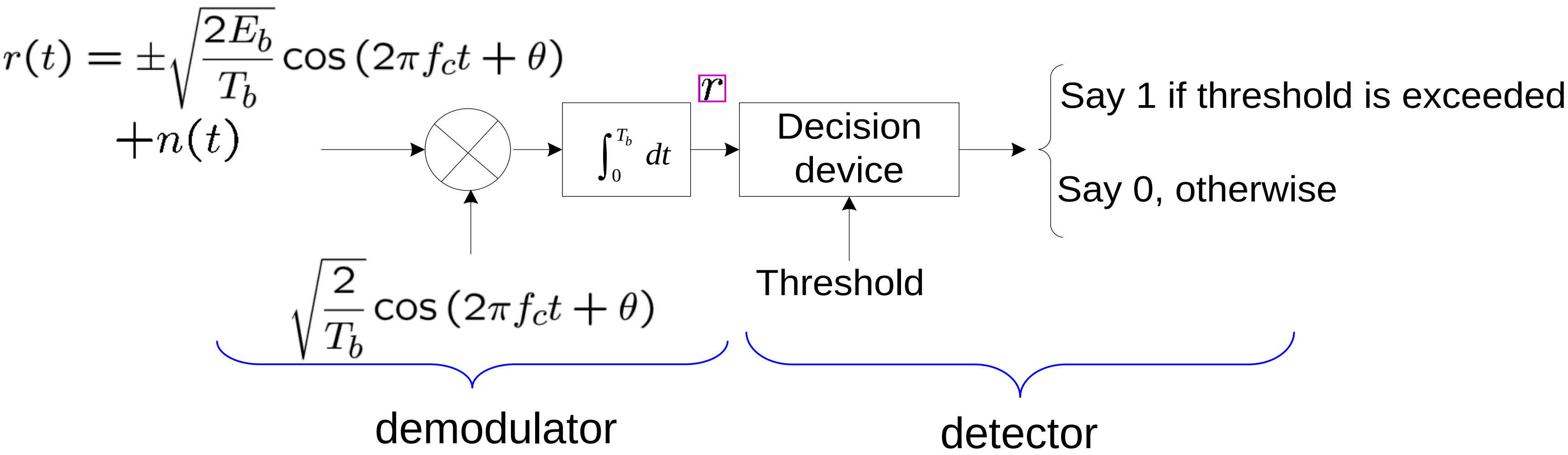
- This ratio is normally called bit energy to noise density ratio (or SNR/bit)

BPSK Transmitter



$$\phi_1(t) = \sqrt{\frac{2}{T_b}} \cos(2\pi f_c t)$$

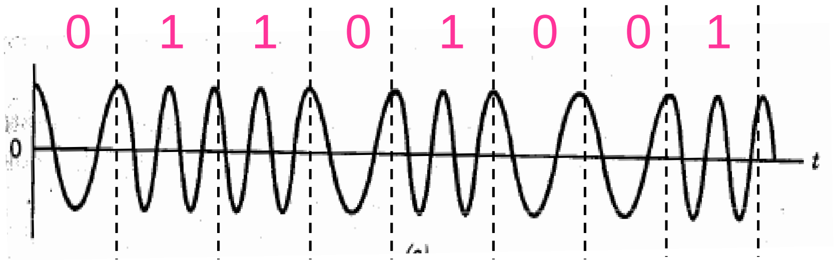
BPSK Receiver



- θ is the carrier-phase offset, due to propagation delay or oscillators at transmitter and receiver are not synchronous
- The detection is coherent in the sense of
  - Phase synchronization
  - Timing synchronization

Binary FSK

- Modulation



"1" → $s_1(t) = \sqrt{\frac{2E_b}{T_b}} \cos(2\pi f_1 t)$

"0" → $s_2(t) = \sqrt{\frac{2E_b}{T_b}} \cos(2\pi f_2 t)$

$0 \le t < T_b$

- E_b : transmitted signal energy per bit

$$\int_0^{T_b} s_1^2(t)dt = \int_0^{T_b} s_2^2(t)dt = E_b$$

- f_i : transmitted frequency with separation $\Delta f = f_1 - f_0$
- Δf is selected so that $s_1(t)$ and $s_2(t)$ are orthogonal i.e.

$$\int_0^{T_b} s_1(t)s_2(t)dt = 0$$

(Example?)

Signal Space for BFSK

- Two orthogonal basis functions are required

$$\phi_1(t) = \sqrt{\frac{2}{T_b}} \cos(2\pi f_1 t) \qquad 0 \le t < T_b$$

$$\phi_2(t) = \sqrt{\frac{2}{T_b}} \cos(2\pi f_2 t) \qquad 0 \le t < T_b$$

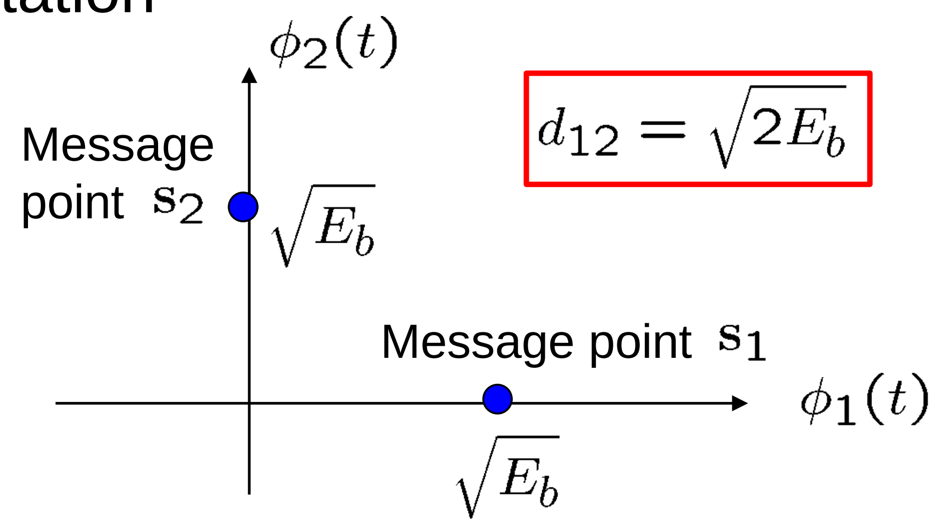
$$\Rightarrow \quad s_1(t) = \sqrt{E_b}\,\phi_1(t)$$

$$s_2(t) = \sqrt{E_b}\,\phi_2(t)$$

- Signal space representation

$$\mathbf{s}_1 = [\sqrt{E_b} \quad 0]$$

$$\mathbf{s}_2 = [0 \quad \sqrt{E_b}]$$



$$d_{12} = \sqrt{2E_b}$$

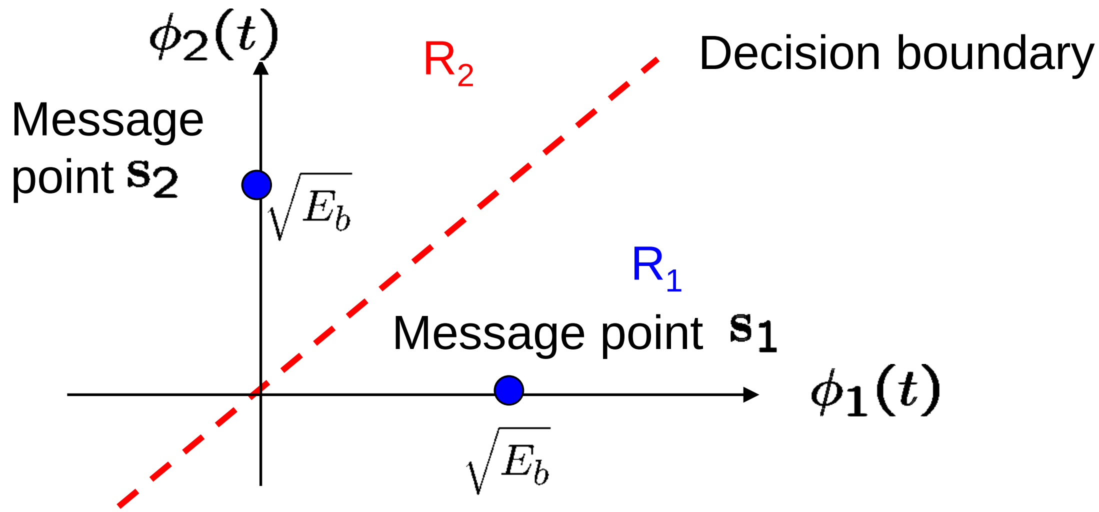
Decision Regions of Binary FSK

- Observation vector

$$\vec{r} = [r_1 \quad r_2]$$

$$r_1 = \int_0^{T_b} r(t)\phi_1(t)dt$$

$$r_2 = \int_0^{T_b} r(t)\phi_2(t)dt$$



- The receiver decides in favor of s_1 if the observation vector r falls inside region R_1. This occurs when $r_1 > r_2$
- When $r_1 < r_2$, r falls inside region R_2 and the receiver decides in favor of s_2

Probability of Error for Binary FSK

- Given that s_1 is transmitted,

$$r_1 = \sqrt{E_b} + n_1 \quad \text{and} \quad r_2 = n_2$$

- Since the condition $r_1 < r_2$ corresponds to the receiver making a decision in favor of symbol s_2, the conditional probability of error given s_1 is transmitted is given by

$$P(e|s_1) = P(r_1 < r_2|s_1) = P(\sqrt{E_b} + n_1 < n_2)$$

- Define a new random variable $n = n_1 - n_2$
- Since n_1 and n_2 are ***i.i.d*** with $n_1, n_2 \in \mathcal{N}(0, N_0/2)$
- Thus, n is also Gaussian with $n \in \mathcal{N}(0, N_0)$

$$\Rightarrow \quad P(e|s_1) = P(n < -\sqrt{E_b}) = Q\left(\sqrt{\frac{E_b}{N_0}}\right)$$

P_e for BFSK (cont'd)

- By symmetry

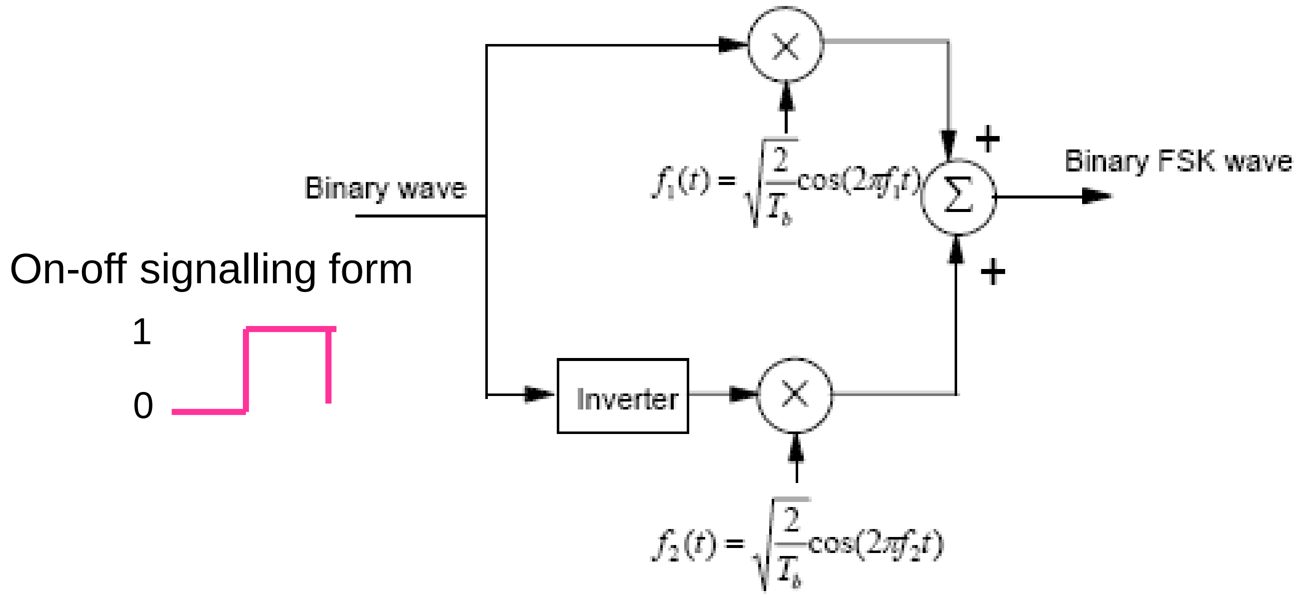
$$P(e|s_2) = P(r_1 > r_2|s_2) = Q\left(\sqrt{\frac{E_b}{N_0}}\right)$$

- Since the two signals are equally likely to be transmitted, the average probability of error for coherent binary FSK is

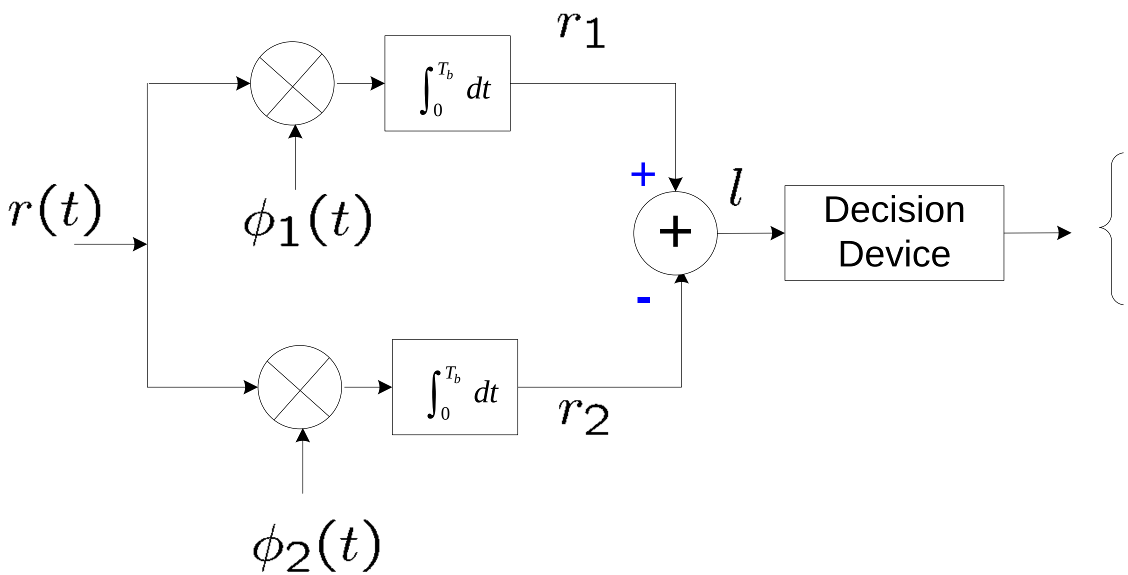
$$P_e = Q\left(\sqrt{\frac{E_b}{N_0}}\right) \Longrightarrow$$ **3 dB** worse than BPSK

To achieve the same P_e, BFSK needs 3dB more transmission power than BPSK

Binary FSK Transmitter



Coherent Binary FSK Receiver



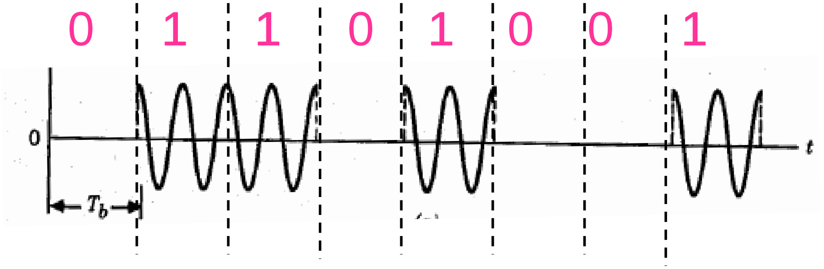
Choose 1 if *l*>0
Choose 0 otherwise

Binary ASK

- Modulation

"1" → $s_1(t) = \sqrt{\frac{2E}{T_b}} \cos(2\pi f_c t)$

"0" → $s_2(t) = 0 \qquad 0 \le t < T_b$



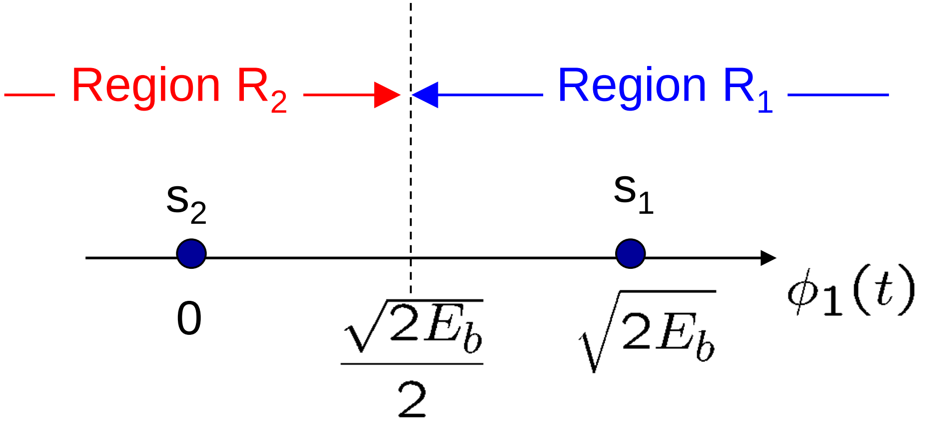
(On-off signaling)

- Average energy per bit

$$E_b = \frac{E+0}{2} \qquad \text{i.e. } E = 2E_b$$

- Decision Region

$$d_{12} = \sqrt{2E_b}$$



Probability of Error for Binary ASK

- Average probability of error is

$$P_e = Q\left(\sqrt{\frac{E_b}{N_0}}\right)$$

Identical to that of coherent binary FSK

- <u>Exercise:</u> Prove P_e

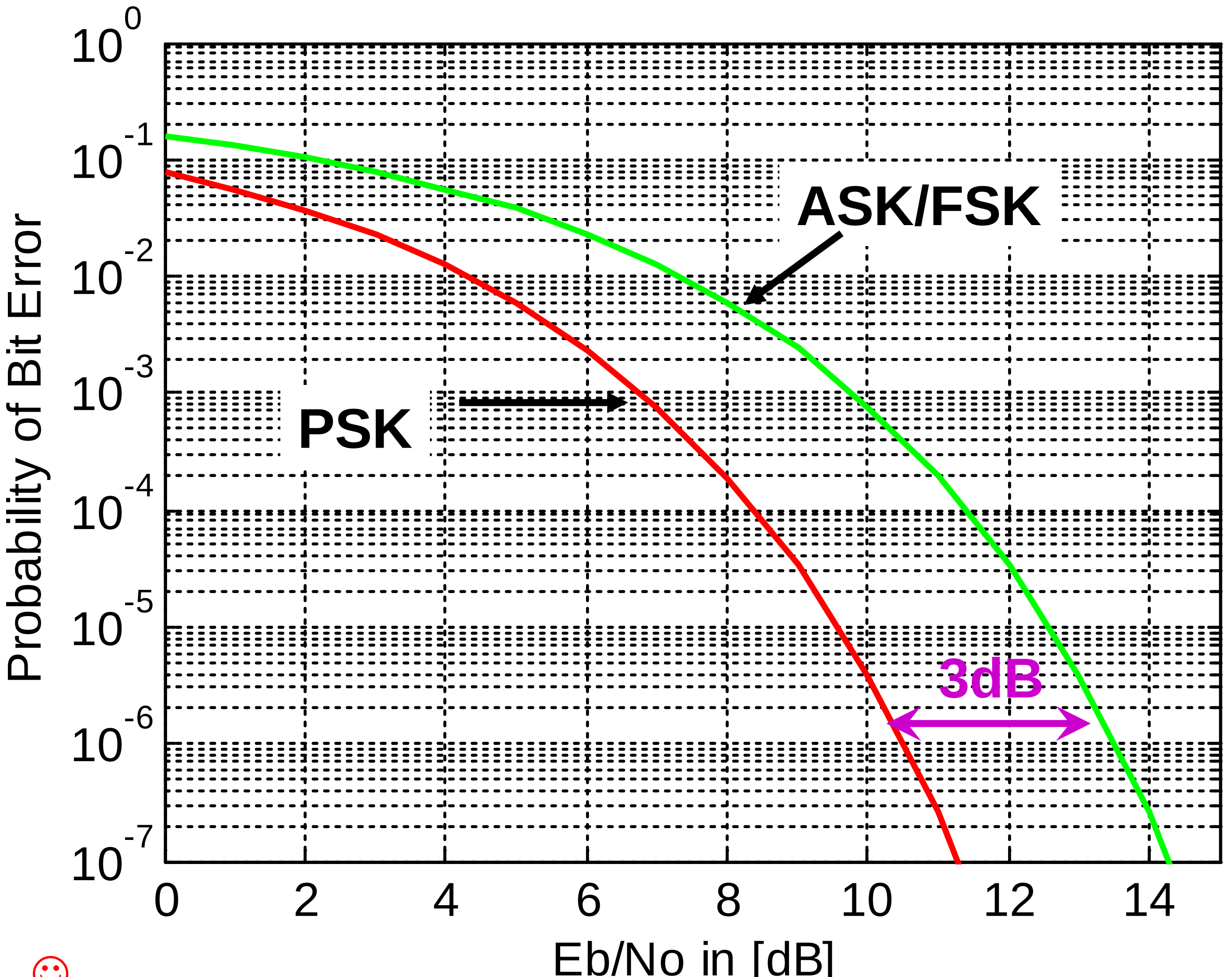
Probability of Error and the Distance Between Signals

BPSK	BFSK	BASK
$d_{1,2} = 2\sqrt{E_b}$	$d_{1,2} = \sqrt{2E_b}$	$d_{1,2} = \sqrt{2E_b}$
$P_e = Q\left(\sqrt{\frac{2E_b}{N_0}}\right)$	$P_e = Q\left(\sqrt{\frac{E_b}{N_0}}\right)$	$P_e = Q\left(\sqrt{\frac{E_b}{N_0}}\right)$

- In general,

$$P_e = Q\left(\sqrt{\frac{d_{12}^2}{2N_0}}\right)$$

Probability of Error for BPSK and FSK/ASK



e.g. ☺

Example

Binary data are transmitted over a microwave link at the rate of 10^6 bits/sec and the PSD of the noise at the receiver input is 10^{-10} watts/Hz.

a) Find the average carrier power required to maintain an average probability of error $P_e \leq 10^{-4}$ for coherent binary FSK.

b) Repeat the calculation in a) for noncoherent binary FSK

- We have discussed
  - Coherent modulation schemes, .e.g. BPSK, BFSK, BASK
  - They needs coherent detection, assuming that the receiver is able to detect and track the carrier wave's phase

- In many practical situations, strict phase synchronization is not possible. In these situations, non-coherent reception is required.
- We now consider:
  - Non-coherent detection on binary FSK
  - Differential phase-shift keying (DPSK)

Non-coherent scheme: BFSK

- Consider a binary FSK system, the two signals are

$$s_1(t) = \sqrt{\frac{2E_b}{T_b}} \cos(2\pi f_1 t + \theta_1)$$

$$s_2(t) = \sqrt{\frac{2E_b}{T_b}} \cos(2\pi f_2 t + \theta_2)$$

$$0 \le t < T_b$$

θ_1, θ_2: unknown random phases with uniform distribution

$$p_{\theta_1}(\theta) = p_{\theta_2}(\theta) = \begin{cases} 1/2\pi & \theta \in [0, 2\pi) \\ 0 & \text{else} \end{cases}$$

Signal Space Representation

- Since

$$s_1(t) = \sqrt{\frac{2E_b}{T_b}} \cos(2\pi f_1 t + \theta_1) = \sqrt{\frac{2E_b}{T_b}} \cos(2\pi f_1 t)\cos(\theta_1) - \sqrt{\frac{2E_b}{T_b}} \sin(2\pi f_1 t)\sin(\theta_1)$$

$$s_2(t) = \sqrt{\frac{2E_b}{T_b}} \cos(2\pi f_2 t + \theta_2) = \sqrt{\frac{2E_b}{T_b}} \cos(2\pi f_2 t)\cos(\theta_2) - \sqrt{\frac{2E_b}{T_b}} \sin(2\pi f_2 t)\sin(\theta_2)$$

- Choose four basis functions as

$$\phi_{1c}(t) = \sqrt{2/T_b}\cos(2\pi f_1 t) \quad \phi_{1s}(t) = -\sqrt{2/T_b}\sin(2\pi f_1 t)$$

$$\phi_{2c}(t) = \sqrt{2/T_b}\cos(2\pi f_2 t) \quad \phi_{2s}(t) = \sqrt{2/T_b}\sin(2\pi f_2 t)$$

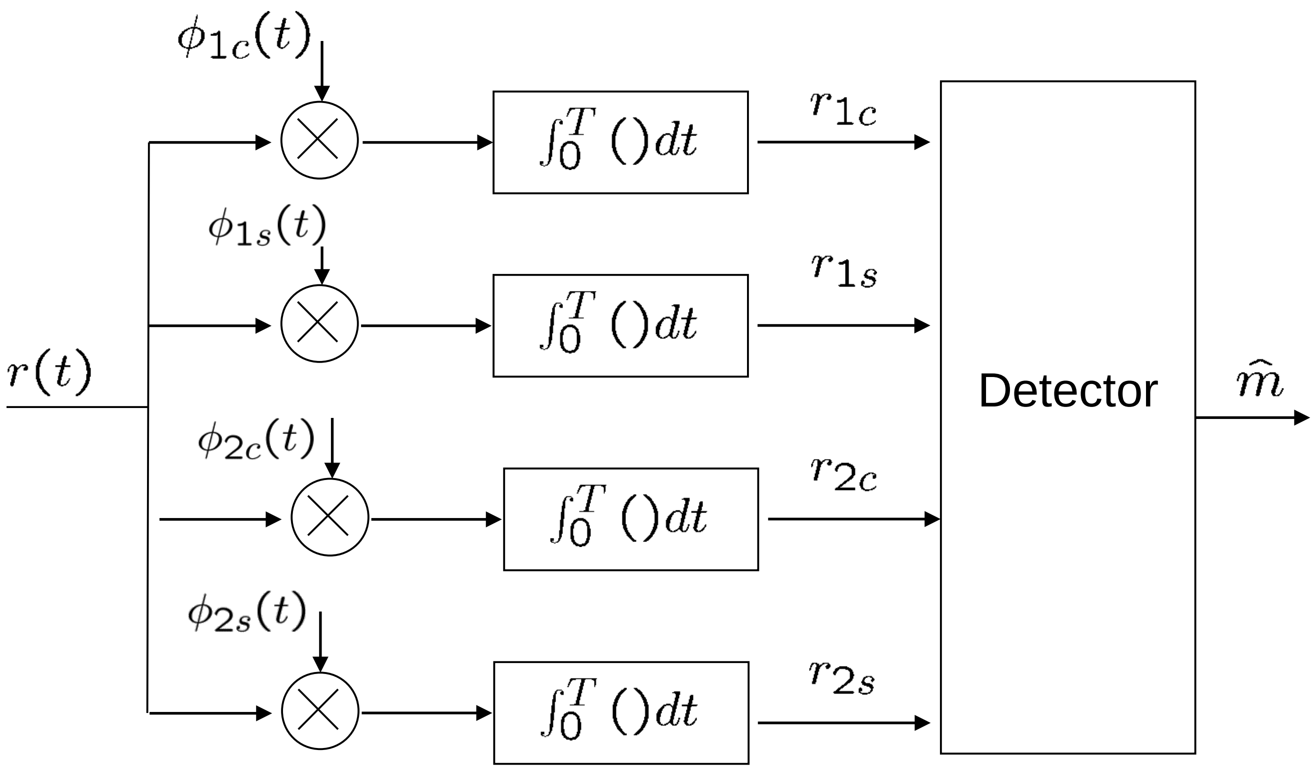
- Signal space representation

$$\vec{s}_1 = [\ \sqrt{E_b}\cos\theta_1 \quad \sqrt{E_b}\sin\theta_1 \quad 0 \quad 0\ ]$$

$$\vec{s}_2 = [\ 0 \quad 0 \quad \sqrt{E_b}\cos\theta_2 \quad \sqrt{E_b}\sin\theta_2\ ]$$

- The vector representation of the received signal

$$\vec{r} = [r_{1c} \quad r_{1s} \quad r_{2c} \quad r_{2s}]$$



Decision Rule for Non-coherent FSK

- ML criterion:

$$f(\vec{r}|\vec{s}_1) \underset{\text{Choose } s_2}{\overset{\text{Choose } s_1}{\gtrless}} f(\vec{r}|\vec{s}_2)$$

- Conditional pdf

$$f(\vec{r}|\vec{s}_1,\theta_1) = \frac{1}{\pi N_0} \exp\left[-\frac{(r_{1c}-\sqrt{E_b}\cos\theta_1)^2+(r_{1s}-\sqrt{E_b}\sin\theta_1)^2}{N_0}\right] \times \frac{1}{\pi N_0} \exp\left[-\frac{r_{2c}^2+r_{2s}^2}{N_0}\right]$$

- Similarly,

$$f(\vec{r}|\vec{s}_2,\theta_2) = \frac{1}{\pi N_0} \exp\left[-\frac{r_{1c}^2+r_{1s}^2}{N_0}\right] \times \frac{1}{\pi N_0} \exp\left[-\frac{(r_{2c}-\sqrt{E_b}\cos\theta_2)^2+(r_{2s}-\sqrt{E_b}\sin\theta_2)^2}{N_0}\right]$$

- For ML decision, we need to evaluate

$$f(\vec{r}|\vec{s}_1) \geq f(\vec{r}|\vec{s}_2)$$

- i.e.

$$\frac{1}{2\pi}\int_0^{2\pi} f(\vec{r}|\vec{s}_1,\theta_1)d\theta_1 \geq \frac{1}{2\pi}\int_0^{2\pi} f(\vec{r}|\vec{s}_2,\theta_2)d\theta_2$$

- Removing the constant terms

$$\left(\frac{1}{\pi N_0}\right)^2 \exp\left[-\frac{{r_{1c}}^2 + {r_{1s}}^2 + {r_{2c}}^2 + {r_{2s}}^2 + E}{N_0}\right]$$

- We have the inequality

$$\frac{1}{2\pi}\int_0^{2\pi}\exp\left[\frac{2\sqrt{E}r_{1c}\cos(\phi_1)+2\sqrt{E}r_{1s}\sin(\phi_1)}{N_0}\right]d\phi_1$$
$$\geq \frac{1}{2\pi}\int_0^{2\pi}\exp\left[\frac{2\sqrt{E}r_{2c}\cos(\phi_1)+2\sqrt{E}r_{2s}\sin(\phi_1)}{N_0}\right]d\phi_2$$

- By definition

$$\frac{1}{2\pi}\int_0^{2\pi}\exp\left[\frac{2\sqrt{E}r_{1c}\cos(\phi_1)+2\sqrt{E}r_{1s}\sin(\phi_1)}{N_0}\right]d\phi_1 = I_0\left(\frac{2\sqrt{E(r_{1c}{}^2+r_{1s}{}^2)}}{N_0}\right)$$

where $I_0(\cdot)$ is a modified Bessel function of the zeroth order

Decision Rule (cont'd)

- Thus, the decision rule becomes: choose s_1 if

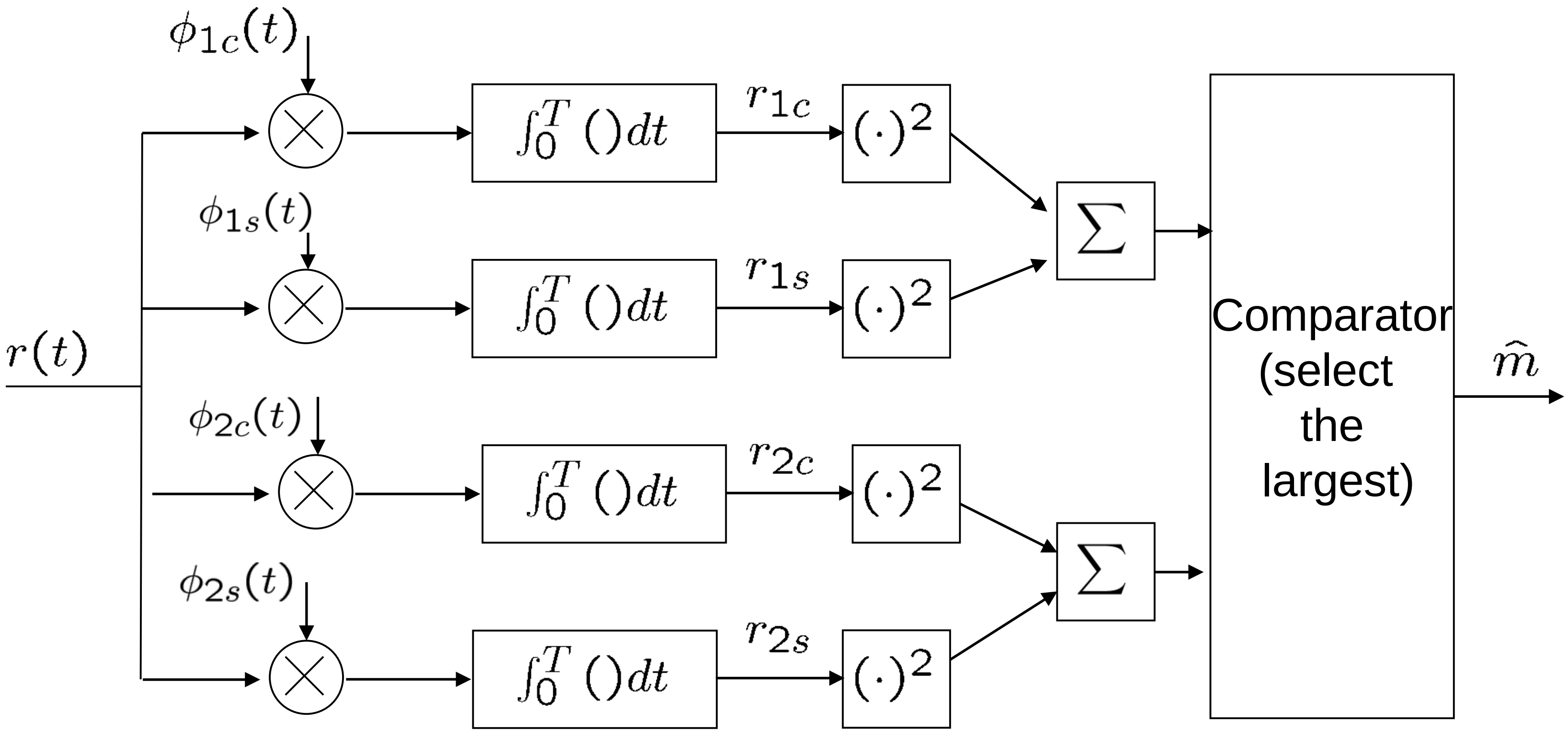
$$I_0\left(\frac{2\sqrt{E(r_{1c}{}^2 + r_{1s}{}^2)}}{N_0}\right) \geq I_0\left(\frac{2\sqrt{E(r_{2c}{}^2 + r_{2s}{}^2)}}{N_0}\right)$$

- Noting that this Bessel function is monotonically increasing. Therefore we choose s_1 if

$$\sqrt{r_{1c}^2 + r_{1s}^2} \geq \sqrt{r_{2c}^2 + r_{2s}^2}$$

- Interpretation: compare the energy in the two frequencies and pick the larger => envelop detector
- Carrier phase is irrelevant in decision making

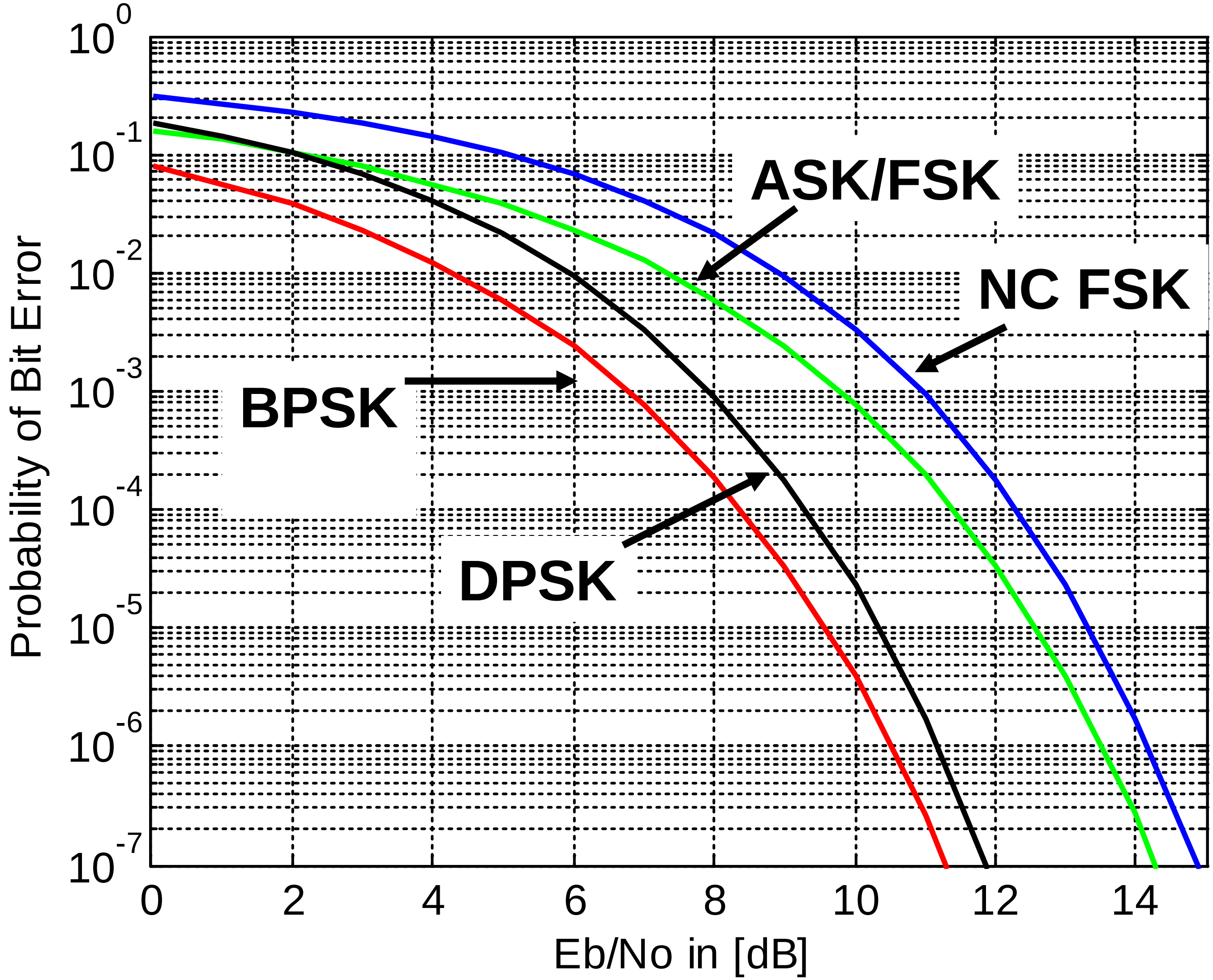
Structure of Non-Coherent Receiver for Binary FSK



- It can be shown that $$P_e = \frac{1}{2}\exp\left(-\frac{E_b}{2N_0}\right)$$

(For detailed proof, see Section 10.4.2 in the textbook)

Performance Comparison Between coherent FSK and Non-Coherent FSK



Differential PSK (DPSK)

- Non-coherent version of PSK
- Phase synchronization is eliminated using differential encoding
  - Encode the information in phase difference between successive signal transmission. In effect,
  - to send "0", advance the phase of the current signal by 180^0;
  - to send "1", leave the phase unchanged
- Provided that the unknown phase θ contained in the received wave varies slowly (constant over two bit intervals), the phase difference between waveforms received in two successive bit intervals will be independent of θ.

Generation of DPSK signal

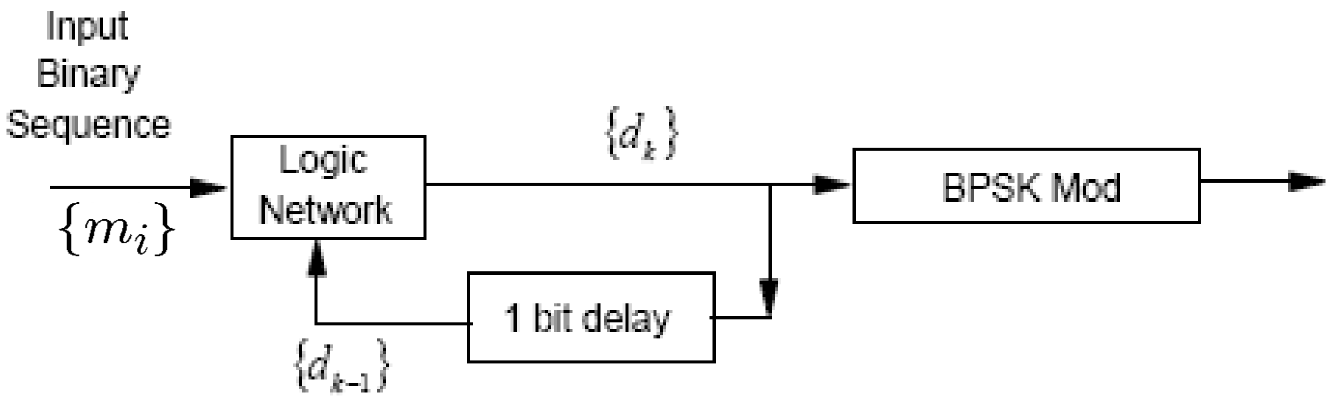
- Generate DPSK signals in two steps
  - Differential encoding of the information binary bits
  - Phase shift keying

- Differential encoding starts with an arbitrary reference bit

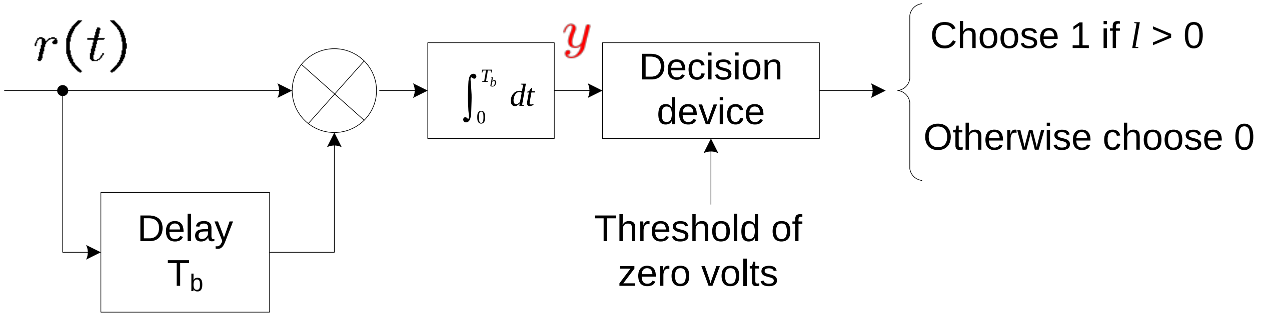
Information sequence		1	0	0	1	0	0	1	1	$\{m_i\}$
Differentially encoded sequence	1 (Initial bit)	1	0	1	1	0	1	1	1	$\{d_i\}$
Transmitted Phase	0	0	π	0	0	π	0	0	0	

$$d_i = \overline{d_{i-1} \oplus m_i}$$

DPSK Transmitter Diagram



Differential Detection of DPSK Signals



- Output of integrator (assume noise free)

$$y = \int_0^{T_b} r(t) r(t - T_b) dt = \int_0^{T_b} \cos(w_c t + \psi_k + \theta) \cos(w_c t + \psi_{k-1} + \theta) dt$$

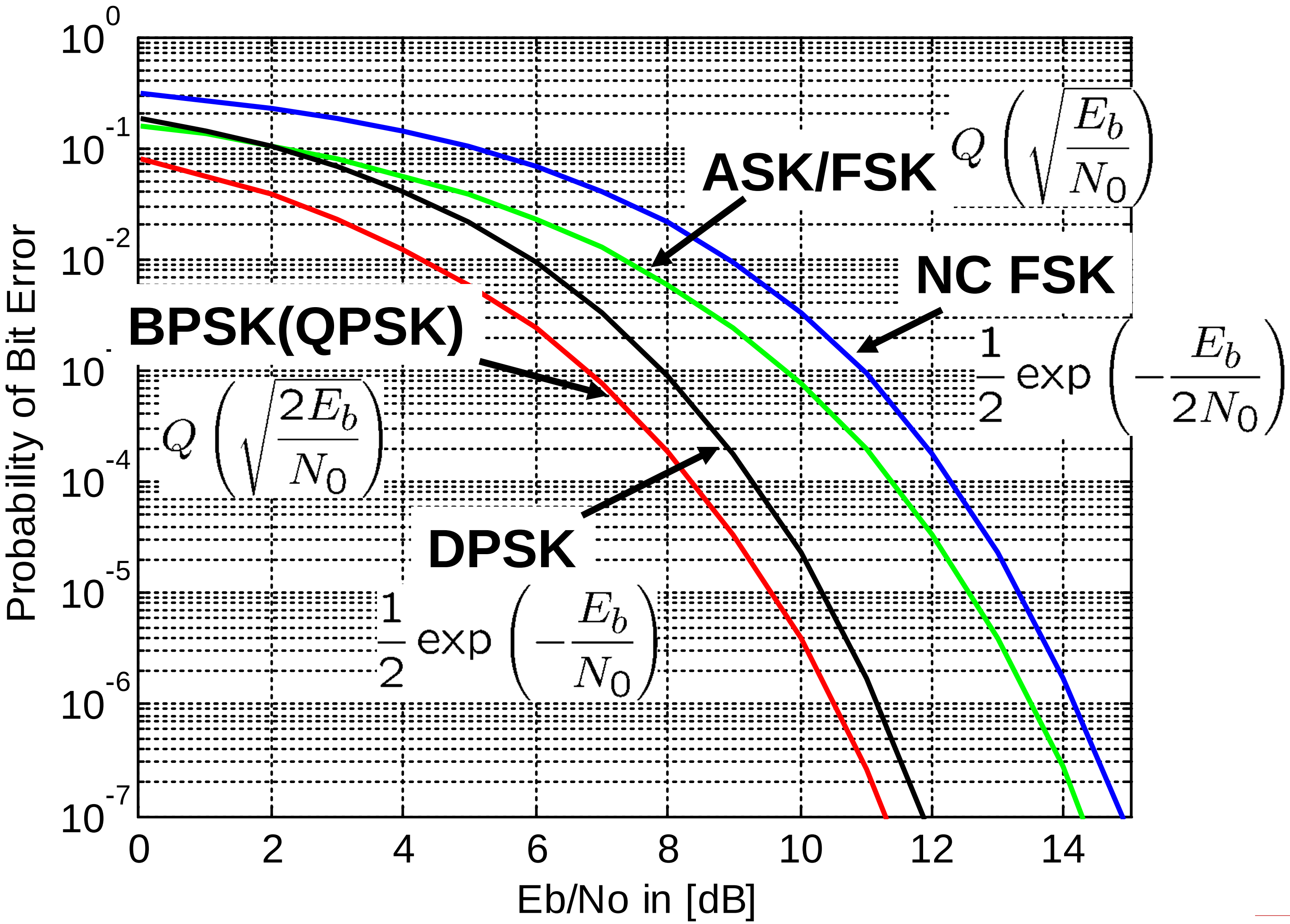
$$\propto \cos(\psi_k - \psi_{k-1})$$

  - The unknown phase θ becomes irrelevant
  - If $\psi_k - \psi_{k-1}$ = 0 (bit 1), then y > 0
  - if $\psi_k - \psi_{k-1}$ =π (bit 0), then y < 0
- Error performance $P_e = \frac{1}{2} \exp\left(-\frac{E_b}{N_0}\right)$

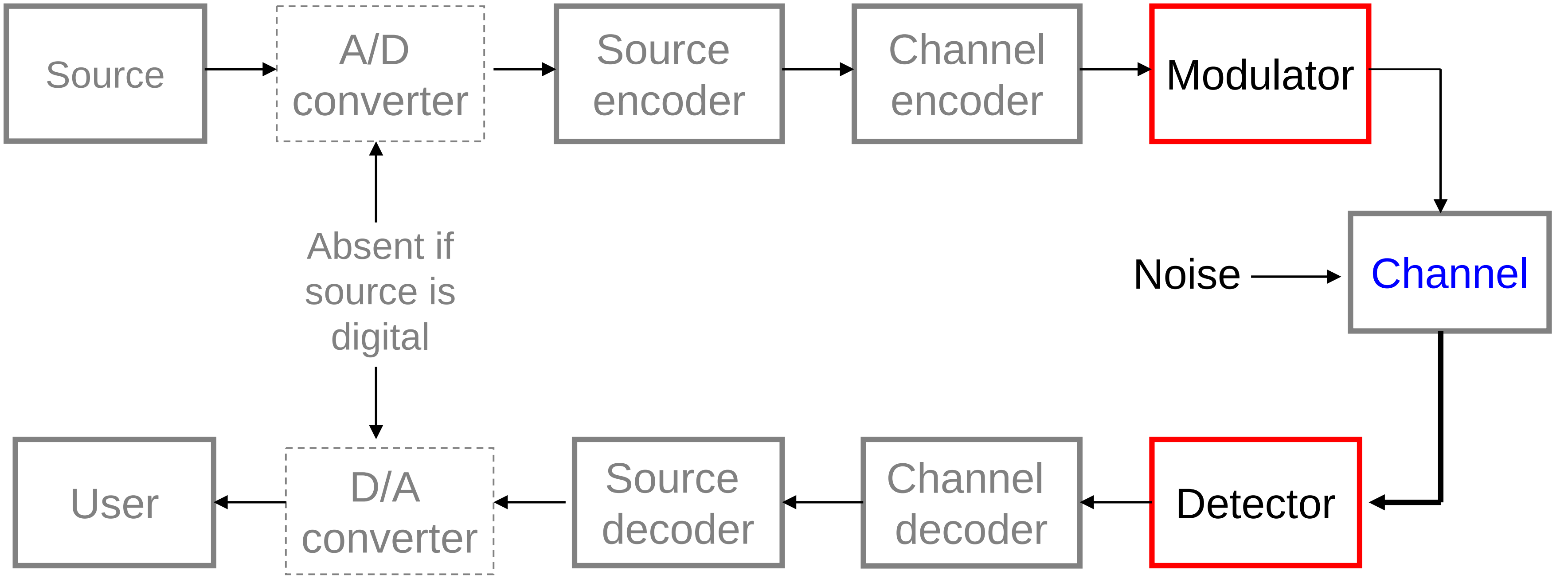
Summary of P_e for Different Binary Modulations

Coherent PSK	$Q\left(\sqrt{\frac{2E_b}{N_0}}\right)$
Coherent ASK	$Q\left(\sqrt{\frac{E_b}{N_0}}\right)$
Coherent FSK	$Q\left(\sqrt{\frac{E_b}{N_0}}\right)$
Non-Coherent FSK	$\frac{1}{2}\exp\left(-\frac{E_b}{2N_0}\right)$
DPSK	$\frac{1}{2}\exp\left(-\frac{E_b}{N_0}\right)$

P_e Plots for Different Binary Modulations



Topics to be Covered



- Binary digital modulation
- M-ary digital modulation
- Comparison study

M-ary Modulation (多进制调制)

256 QAM

128 QAM

64QAM

M-ary Modulation Techniques

- In binary data transmission, send only one of two possible signals during each bit interval T_b
- In M-ary data transmission, send one of M possible signals during each signaling interval T
- In almost all applications, $M = 2^n$ and $T = nT_b$, where n is an integer
- Each of the M signals is called a symbol
- These signals are generated by changing the amplitude, phase, frequency, or combined forms of a carrier in M discrete steps.
- Thus, we have:
  - MASK MPSK MFSK MQAM

M-ary Phase-Shift Keying (MPSK)

- The phase of the carrier takes on M possible values:

$$\theta_m = 2\pi(m-1)/M, \ \ m = 1, \ldots, M$$

- Signal set:

$$s_m(t) = \sqrt{\frac{2E_s}{T}} \cos\left[2\pi f_c t + \frac{2\pi(m-1)}{M}\right] \qquad \begin{array}{c} m = 1, \ldots, M \\ 0 \le t < T \end{array}$$

  - E_s = Energy per symbol
  - $f_c >> \frac{1}{T}$
- Basis functions

$$\phi_1(t) = \sqrt{\frac{2}{T}} \cos(2\pi f_c t)$$

$$0 \le t < T$$

$$\phi_2(t) = \sqrt{\frac{2}{T}} \sin(2\pi f_c t)$$

MPSK (cont'd)

- Signal space representation

$$s_m(t) = \sqrt{\frac{2E_s}{T}} \cos\left[2\pi f_c t + \frac{2\pi(m-1)}{M}\right]$$

$$= \sqrt{\frac{2E_s}{T}} \cos(2\pi f_c t) \cos\left[\frac{2\pi(m-1)}{M}\right]$$

$$-\sqrt{\frac{2E_s}{T}} \sin(2\pi f_c t) \sin\left[\frac{2\pi(m-1)}{M}\right]$$

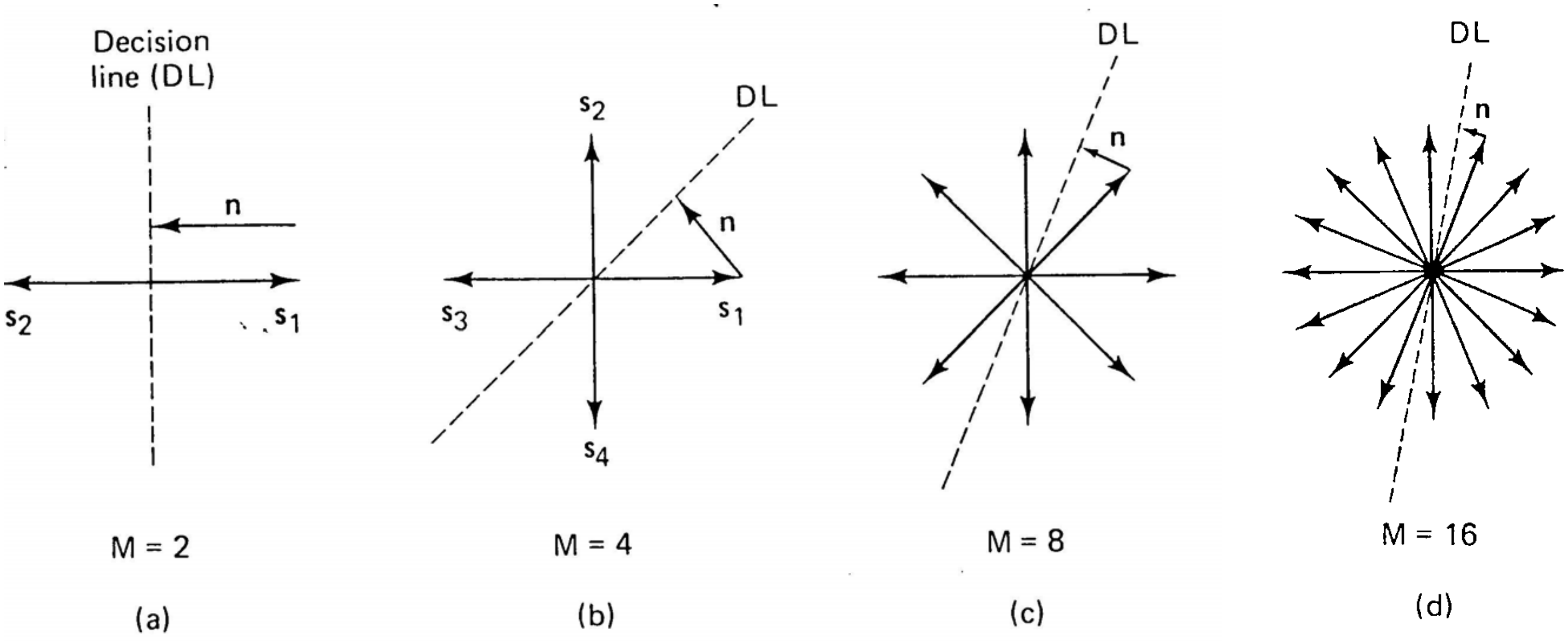
$$= \sqrt{E_s} \cos\left[\frac{2\pi(m-1)}{M}\right]\phi_1(t) - \sqrt{E_s} \sin\left[\frac{2\pi(m-1)}{M}\right]\phi_2(t)$$

⇒

$$\mathbf{s}_m = \left[\ \sqrt{E_s}\cos\left(\frac{2\pi(m-1)}{M}\right) \quad \sqrt{E_s}\sin\left(\frac{2\pi(m-1)}{M}\right)\ \right]$$

$$m = 1, \ldots, M$$

MPSK Signal Constellations



BPSK QPSK 8PSK 16PSK

- Euclidean distance

$$d_{mn} = \|\mathbf{s}_m - \mathbf{s}_n\| = \sqrt{2E_s\left(1-\cos\frac{2\pi(m-n)}{M}\right)}$$

- The minimum Euclidean distance is

$$d_{\min} = \sqrt{2E_s\left(1-\cos\frac{2\pi}{M}\right)} = 2\sqrt{E_s}\sin\frac{\pi}{M}$$

- $d_{\min}$ plays an important role in determining error performance as discussed previously (union bound)
- In the case of PSK modulation, the error probability is dominated by the erroneous selection of either one of the two signal points adjacent to the transmitted signal point.
- Consequently, an approximation to the symbol error probability is

$$P_{MPSK} \approx 2Q\left(\frac{d_{\min}/2}{\sqrt{N_0/2}}\right) = 2Q\left(\sqrt{\frac{2E_s}{N_0}}\sin\frac{\pi}{M}\right)$$

Exercise

- Consider the M=2, 4, 8 PSK signal constellations. All have the same transmitted signal energy Es.
- Determine the minimum distance $d_{\min}$ between adjacent signal points
- For M=8, determine by how many dB the transmitted signal energy Es must be increased to achieve the same $d_{\min}$ as M =4.

Error Performance of MPSK

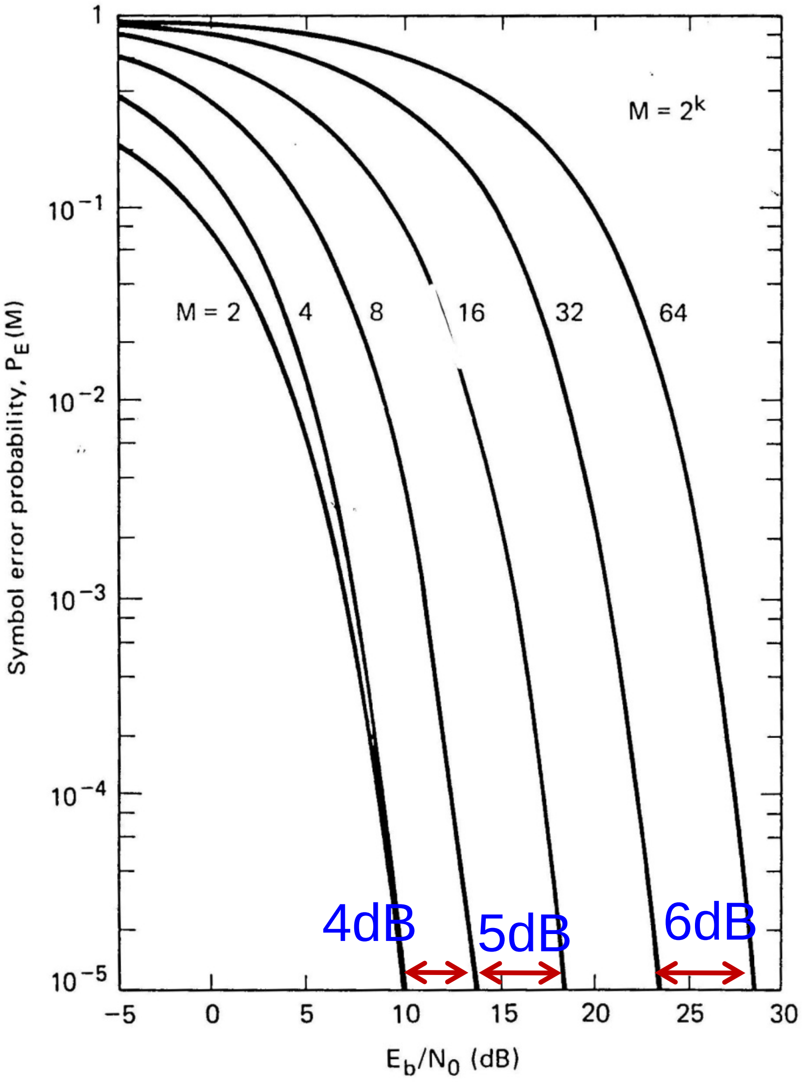


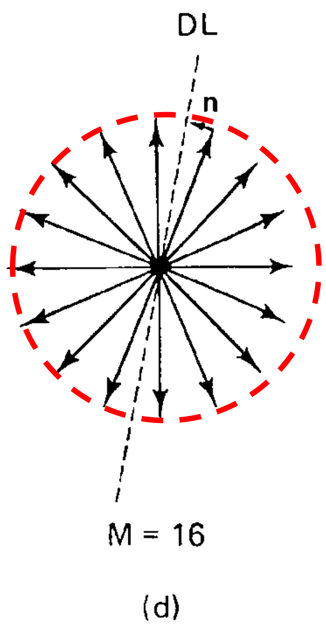
Figure 3.32 Symbol error probability for coherently detected multiple phase signaling. (Reprinted from W. C. Lindsey and M. K. Simon, *Telecommunication Systems Engineering*, Prentice-Hall, Inc., Englewood Cliffs, N.J., 1973, courtesy of W. C. Lindsey and Marvin K. Simon.)

- For large M, doubling the number of phases requires an additional 6dB/bit to achieve the same performance

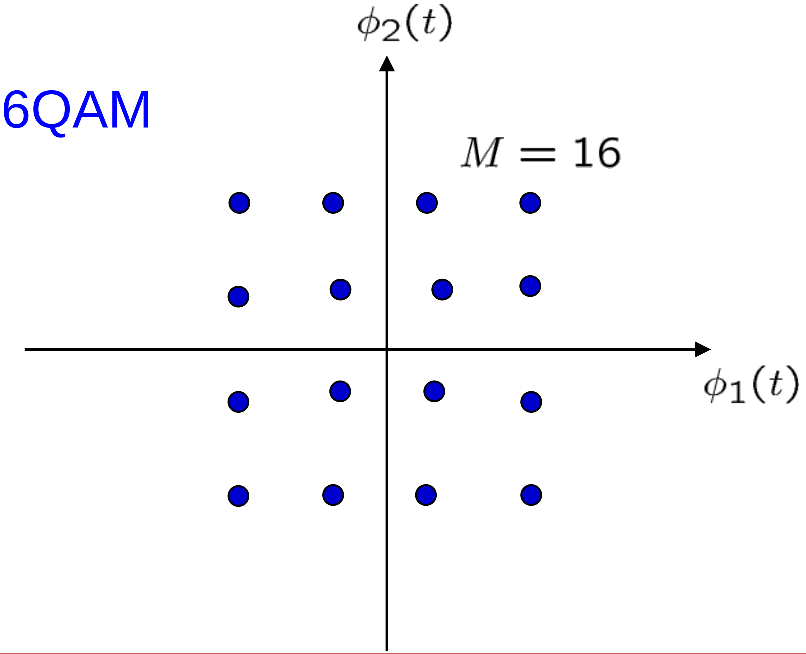
M-ary Quadrature Amplitude Modulation (MQAM 正交幅度调制)

- In MPSK, in-phase and quadrature components are interrelated in such a way that the envelope is constant (circular constellation)
- If we relax this constraint, we get M-ary QAM

16PSK



16QAM



MQAM

- Signal set:

$$s_i(t) = \sqrt{\frac{2E_0}{T}} a_i \cos(2\pi f_c t) + \sqrt{\frac{2E_0}{T}} b_i \sin(2\pi f_c t) \qquad 0 \leq t < T$$

  - E_0 is the energy of the signal with the lowest amplitude
  - a_i, b_i are a pair of independent integers

- Basis functions:

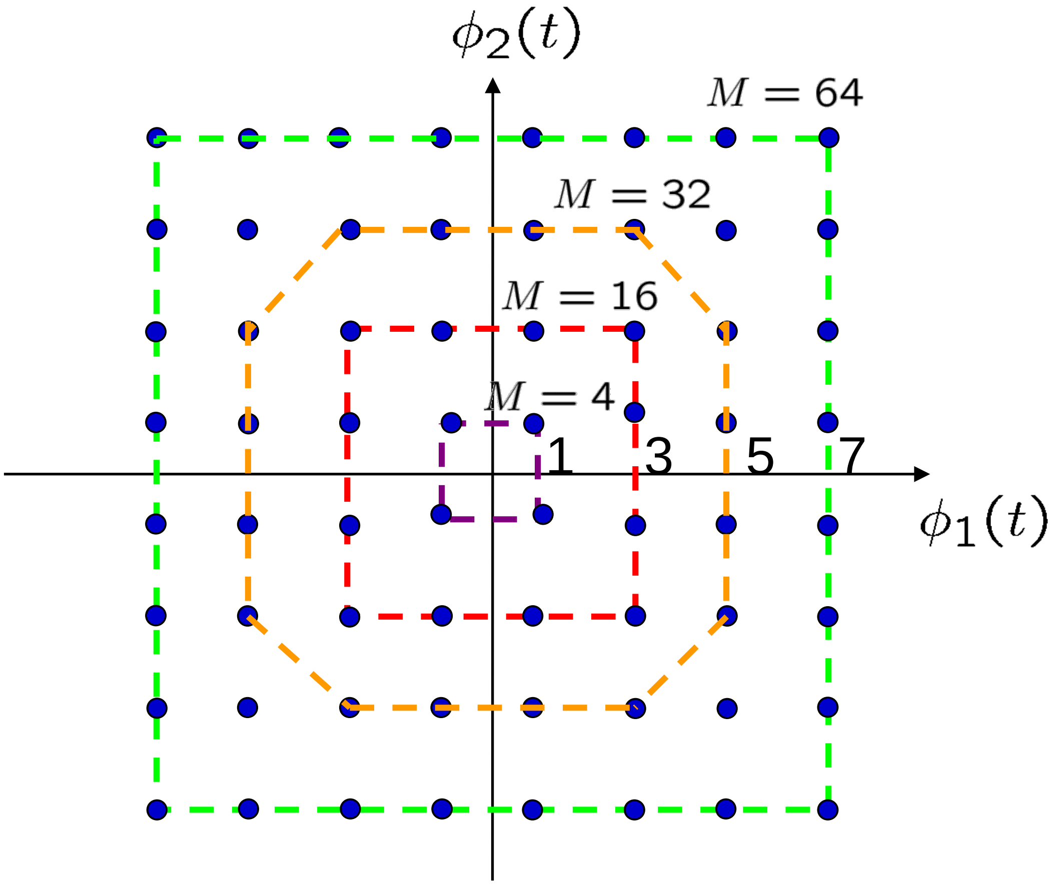
$$\phi_1(t) = \sqrt{\frac{2}{T}} \cos(2\pi f_c t) \quad \phi_2(t) = \sqrt{\frac{2}{T}} \sin(2\pi f_c t) \quad 0 \leq t < T$$

- Signal space representation

$$\vec{s}_i = [\sqrt{E_0} a_i \quad \sqrt{E_0} b_i]$$

MQAM Signal Constellation

- Square lattice



Error Performance of MQAM

- Upper bound of the symbol error probability

$$P_e \le 4Q\left(\sqrt{\frac{3kE_b}{(M-1)N_0}}\right) \qquad \text{(for } M = 2^k \text{)}$$

- **Exercise:**

  Determine the increase in Eb required to maintain the same error performance if the number of bits per symbol is increased from k to k+1, where k is large.

M-ary Frequency-Shift Keying (MFSK) or Multitone Signaling

- Signal set:

$$s_m(t) = \sqrt{\frac{2E_s}{T}} \cos\{2\pi(f_c + (m-1)\triangle f)t\} \qquad \begin{matrix} m = 1, \ldots, M \\ 0 \le t < T \end{matrix}$$

where $\triangle f = f_m - f_{m-1}$ with $f_m = f_c + m\triangle f$

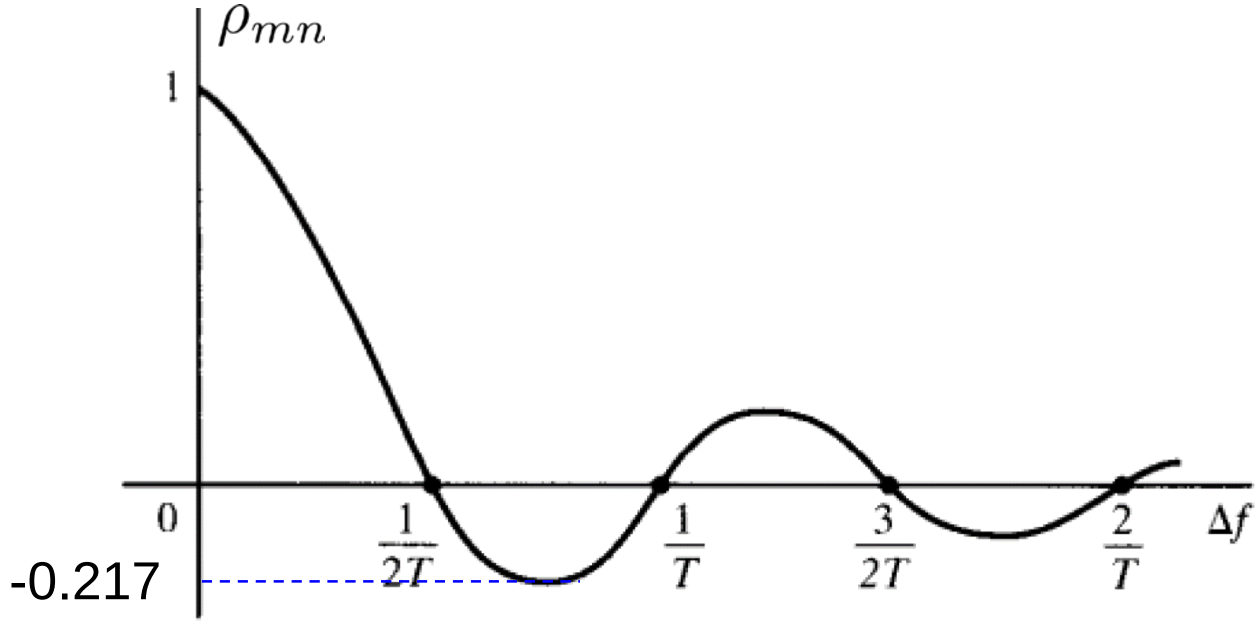
- Correlation between two symbols

$$\rho_{mn} = \frac{1}{E_s}\int_0^T s_m(t)s_n(t)dt$$

$$= \frac{\sin[2\pi(m-n)\triangle fT]}{2\pi(m-n)\triangle fT}$$

$$= \text{sinc}[2(m-n)\triangle fT]$$

MFSK (cont'd)



- For orthogonality, the minimum frequency separation is

$$\Delta f = \frac{1}{2T}$$

- M-ary orthogonal FSK has a geometric presenation as M M-dim orthogonal vectors, given as

$$\mathbf{s}_0 = \left(\sqrt{E_s}, 0, 0, \cdots, 0\right)$$

$$\mathbf{s}_1 = \left(0, \sqrt{E_s}, 0, \cdots, 0\right)$$

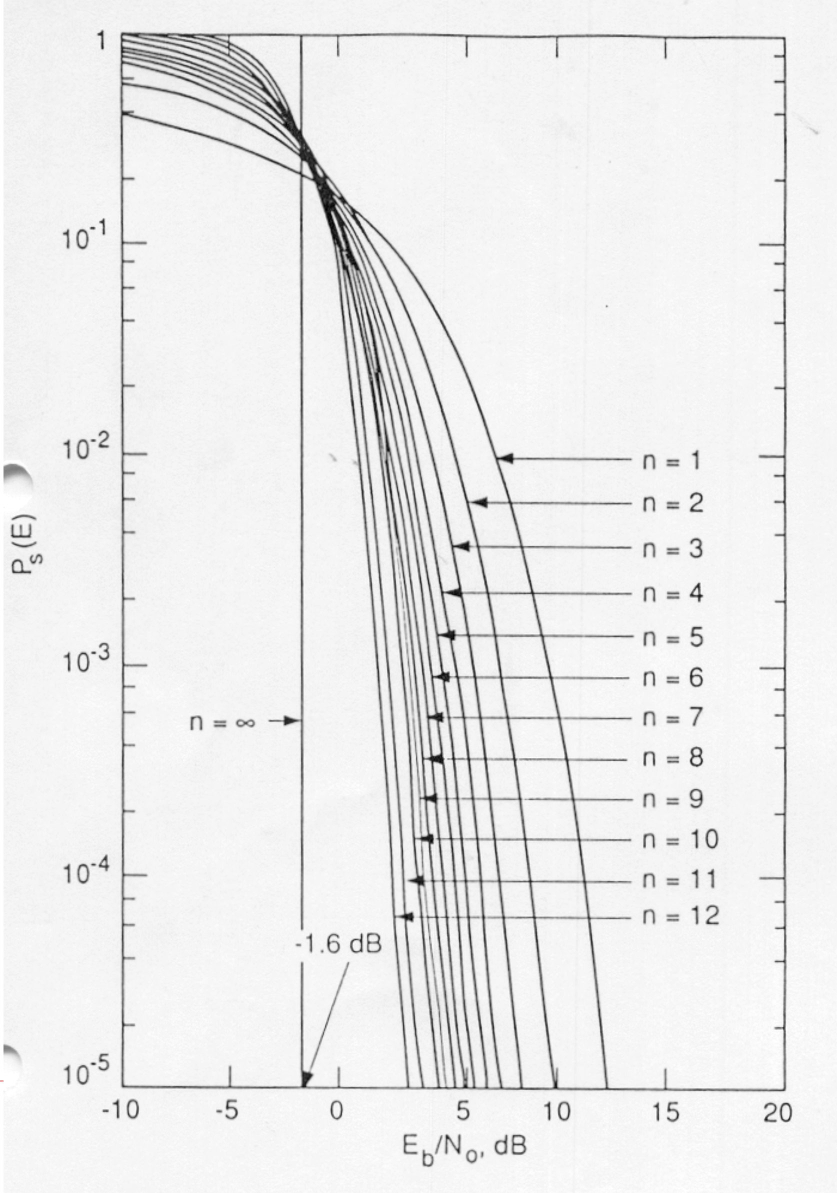
$$\vdots$$

$$\mathbf{s}_{M-1} = \left(0, 0, \cdots, 0, \sqrt{E_s}\right)$$

- The basis functions are

$$\phi_m = \sqrt{\frac{2}{T}} \cos 2\pi \left(f_c + m\Delta f \right) t$$

Error Performance of MFSK



Notes on Error Probability Calculations

- Pe is found by integrating conditional probability of error over the decision region
  - Difficult for multi-dimensions
  - Can be simplified using union bound (see ch07)

- Pe depends only on the distance profile of signal constellation

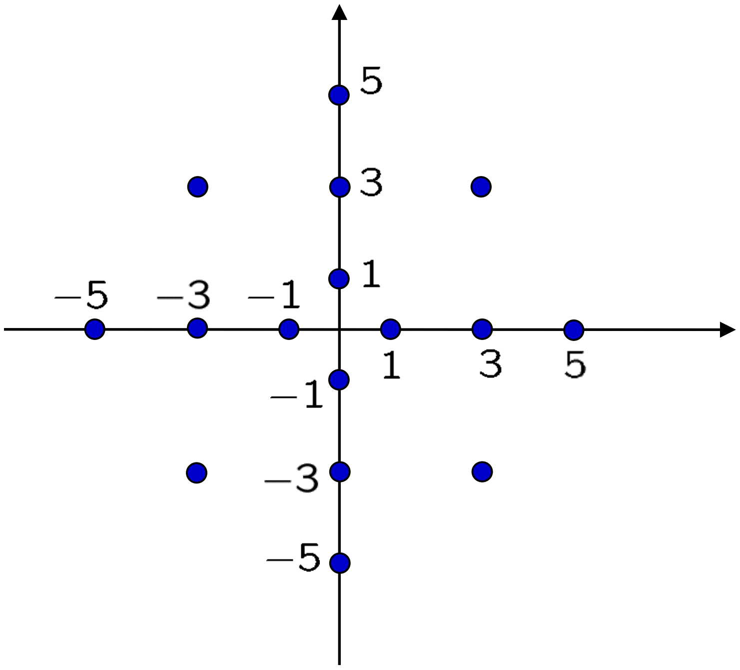
Example

The 16-QAM signal constellation shown below is an international standard for telephone-line modems (called V.29).

a) Determine the optimum decision boundaries for the detector

b) Derive the union bound of the probability of symbol error assuming that the SNR is sufficiently high so that errors only occur between adjacent points

c) Specify a Gray code for this 16-QAM V.29 signal constellation



Symbol Error versus Bit Error

- Symbol errors are different from bit errors
- When a symbol error occurs, all k bits could be in error
- In general, we can find BER using

$$P_b = \sum_{i=1}^{M} P(\vec{s}_i) \sum_{j=1, j\neq i}^{M} \frac{n_{i,j}}{\log_2 M} P\left(\hat{\vec{s}} = \vec{s}_j | \vec{s}_i\right)$$

  - n_{ij} is the number different bits between s_i and s_j
- Gray coding is a bit-to-symbol mapping, where two adjacent symbols differ in only one bit out of the k bits
- An error between adjacent symbol pairs results in one and only one bit error.

Example: Gray Code for QPSK

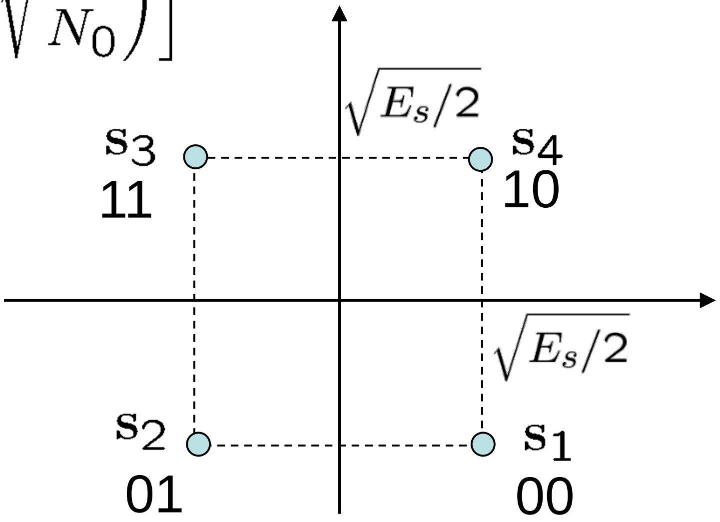
$$P_b = \sum_{i=1}^{M} \frac{1}{4} \sum_{j=1, j\neq i}^{M} \frac{n_{i,j}}{\log_2 M} P\left(\hat{\vec{s}} = \vec{s}_j | \vec{s}_i\right)$$

$$= \frac{1}{2} P\left(\hat{\vec{s}} = \vec{s}_1 | \vec{s}_4\right) + \frac{2}{2} P\left(\hat{\vec{s}} = \vec{s}_2 | \vec{s}_4\right) + \frac{1}{2} P\left(\hat{\vec{s}} = \vec{s}_3 | \vec{s}_4\right)$$

$$= \left[1 - Q\left(\sqrt{\frac{E_s}{N_0}}\right)\right] \cdot Q\left(\sqrt{\frac{E_s}{N_0}}\right) + \left[Q\left(\sqrt{\frac{E_s}{N_0}}\right)\right]^2$$

$$= Q\left(\sqrt{\frac{E_s}{N_0}}\right)$$

$$= Q\left(\sqrt{\frac{2E_b}{N_0}}\right)$$



Bit Error Rate for MPSK and MFSK

- For MPSK with Gray coding
  - An error between adjacent symbols will most likely occur
  - Thus, bit error probability can be approximated by

$$P_b \approx \frac{P_e}{\log_2 M}$$

- For MFSK
  - When an error occurs anyone of the other symbols may result equally likely.
  - Thus, k/2 bits every k bits will on average be in error when there is a symbol error
  - Bit error rate is approximately half of the symbol error rate

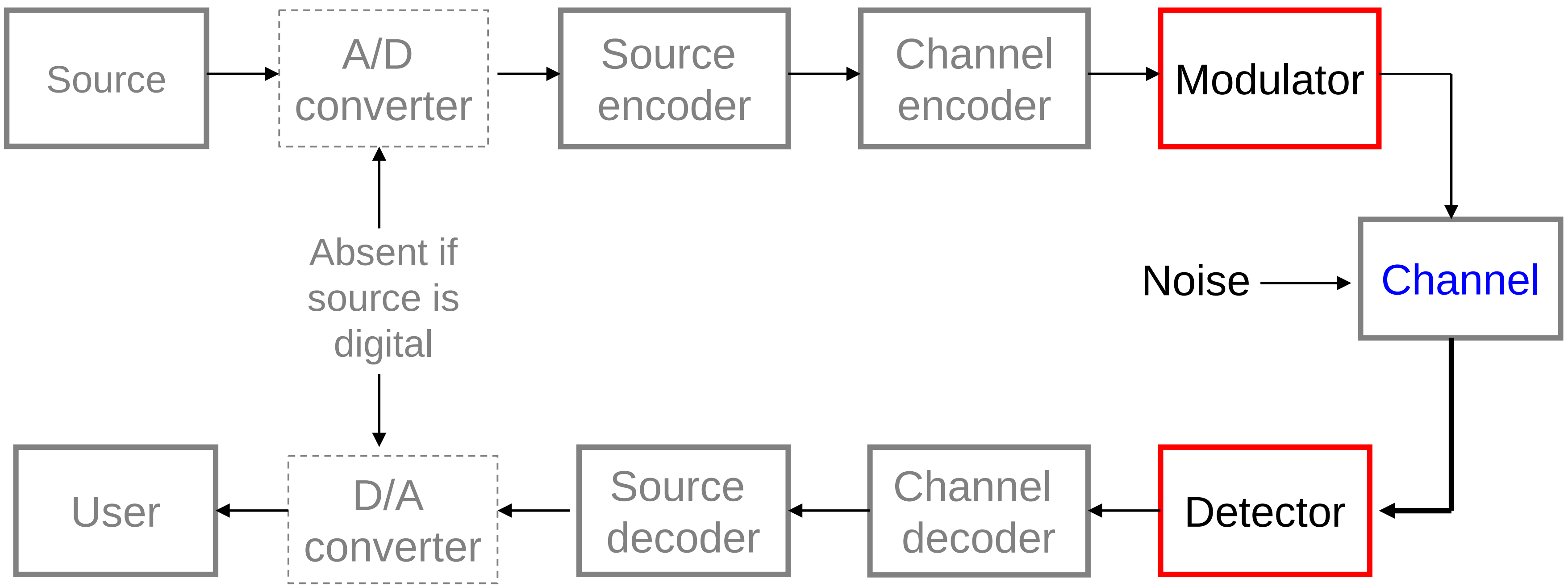
$$P_b \cong \frac{1}{2} P_e$$

Think …

- Why 4G LTE prefers MQAM over MPSK/MFSK?

256QAM

Topics to be Covered



- Binary digital modulation
- M-ary digital modulation
- Comparison study

Comparison of M-ary Modulation Techniques

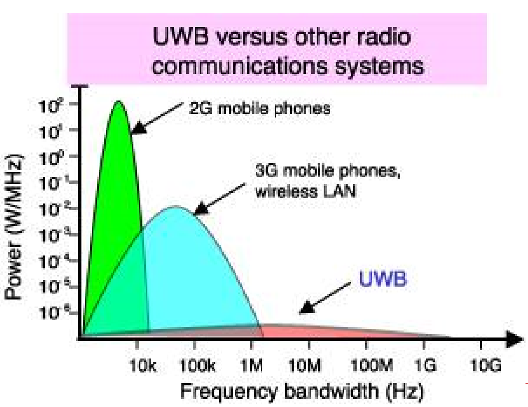
- Channel bandwidth and transmit power are two primary communication resources and have to be used as efficient as possible
  - Power utilization efficiency (energy efficiency): measured by the required E_b/N_o to achieve a certain bit error probability
  - Spectrum utilization efficiency (bandwidth efficiency): measured by the achievable data rate per unit bandwidth R_b/B
- It is always desired to maximize bandwidth efficiency at a minimal required Eb/No

Example

- Suppose you are a system engineer in Huawei/ZTE, designing a part of the communication systems. You are required to design a modulation scheme for three systems using MFSK, MPSK or MQAM only. State the modulation level M to be low, medium or high

An ultra-wideband system

- Large amount of bandwidth
- Band overlays with other systems
- Purpose: high data rate



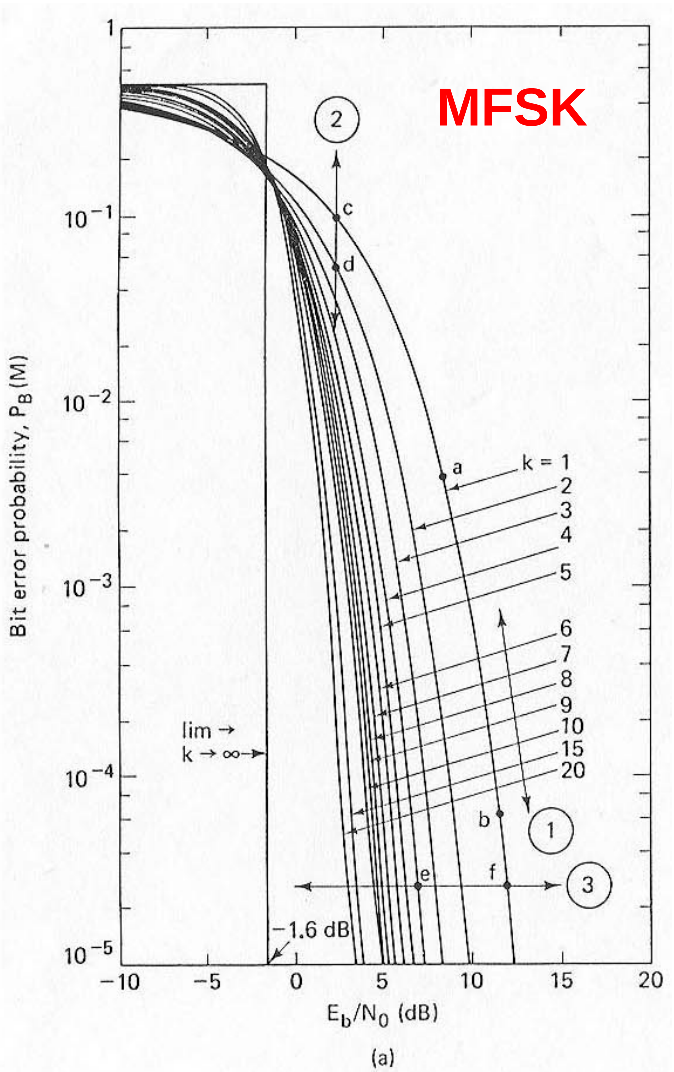
A wireless remote control system

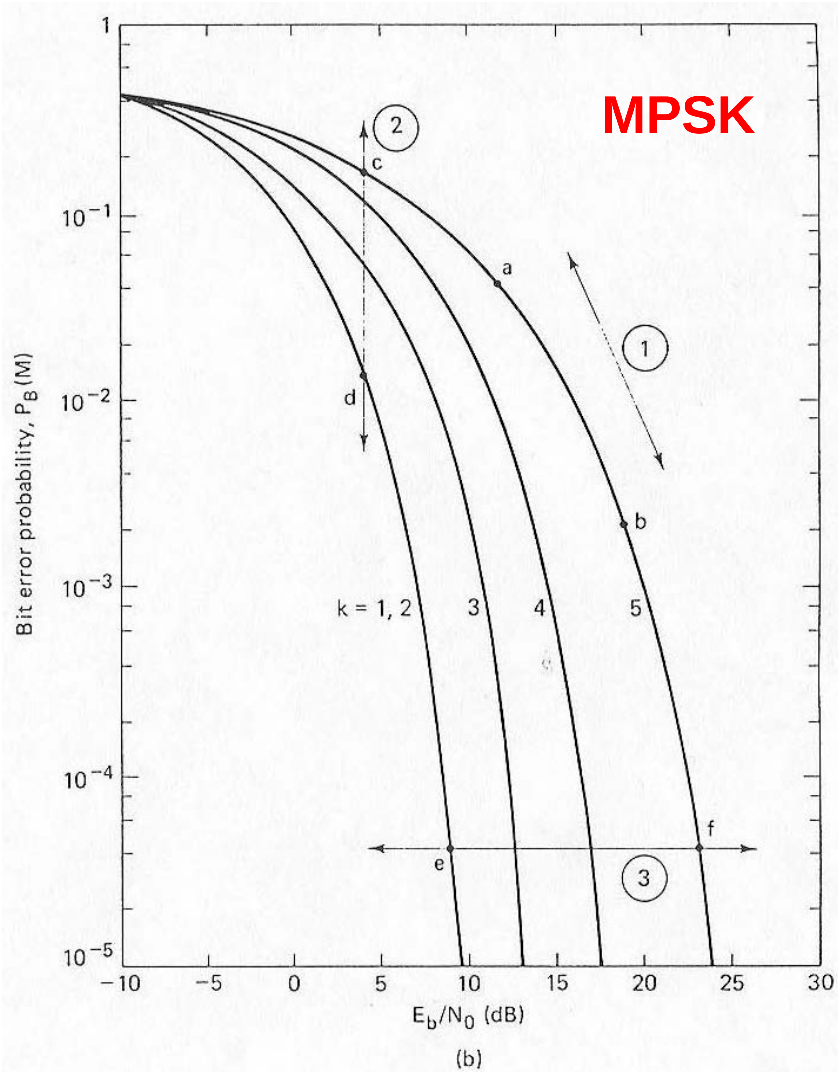
- Use unlicensed band
- Purpose: control devices remotely

A fixed wireless system

- Use licensed band
- Transmitter and receiver fixed with power supply
- Voice and data connections in rural areas

Energy Efficiency Comparison





Picture from: Proakis J G, Salehi M. Fundamentals of communication systems[M]. Pearson Education India, 2007.

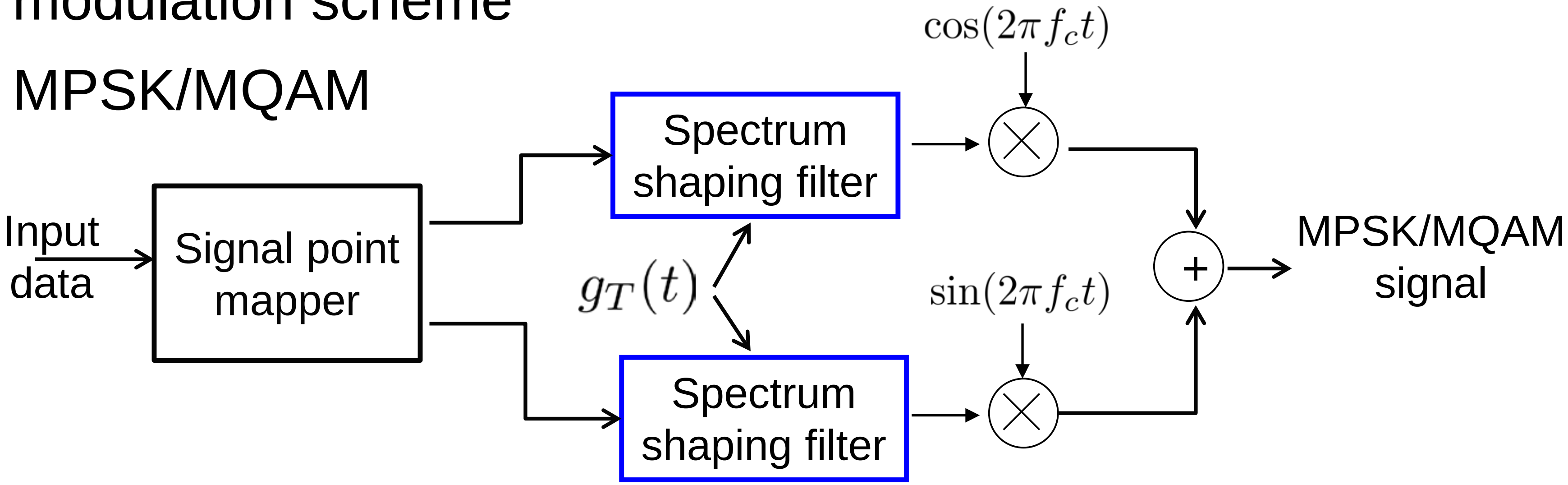
Energy Efficiency Comparison (cont'd)

- MFSK:
  - At fixed E_b/N_o, increase M can provide an improvement on P_b
  - At fixed P_b increase M can provide a reduction in the E_b/N_o requirement
- MPSK
  - BPSK and QPSK have the same energy efficiency
  - At fixed E_b/N_o, increase M degrades Pb
  - At fixed Pb, increase M increases the Eb/No requirement

MFSK is more energy efficient than MPSK

Bandwidth Efficiency Comparison

- To compare bandwidth efficiency, we need to know the power spectral density (power spectra) of a given modulation scheme
- MPSK/MQAM



  - If $g_T(t)$ is rectangular, the bandwidth of mainlope is $B = \frac{2}{T_s}$
  - If it has a raised cosine spectrum, the bandwidth is

$$B = \frac{1+\alpha}{T_s}$$

Bandwidth Efficiency Comparison (cont'd)

- In general, bandwidth required to pass MPSK/MQAM signal is approximately given by

$$B = \frac{1}{T_s}$$

- But

$$R_b = \frac{\log_2 M}{T_s} \text{ = bit rate}$$

- Then bandwidth efficiency may be expressed as

$$\rho = \frac{R_b}{B} = \log_2 M \quad \text{(bits/sec/Hz)}$$

Bandwidth Efficiency Comparison (cont'd)

- MFSK:
  - Bandwidth required to transmit MFSK signal is

$$B = \frac{M}{2T}$$

(Adjacent frequencies need to be separated by 1/2T to maintain orthogonality)

  - Bandwidth efficiency of MFSK signal

$$\rho = \frac{R_b}{B} = \frac{2\log_2 M}{M} \quad \text{(bits/s/Hz)}$$

M	2	4	8	16	32	64
ρ (bits/s/Hz)	1	1	0.75	0.5	0.3125	0.1875

As M increases, bandwidth efficiency of MPSK/MQAM increases, but bandwidth efficiency of MFSK decreases.

Fundamental Tradeoff :
Bandwidth Efficiency and Energy Efficiency

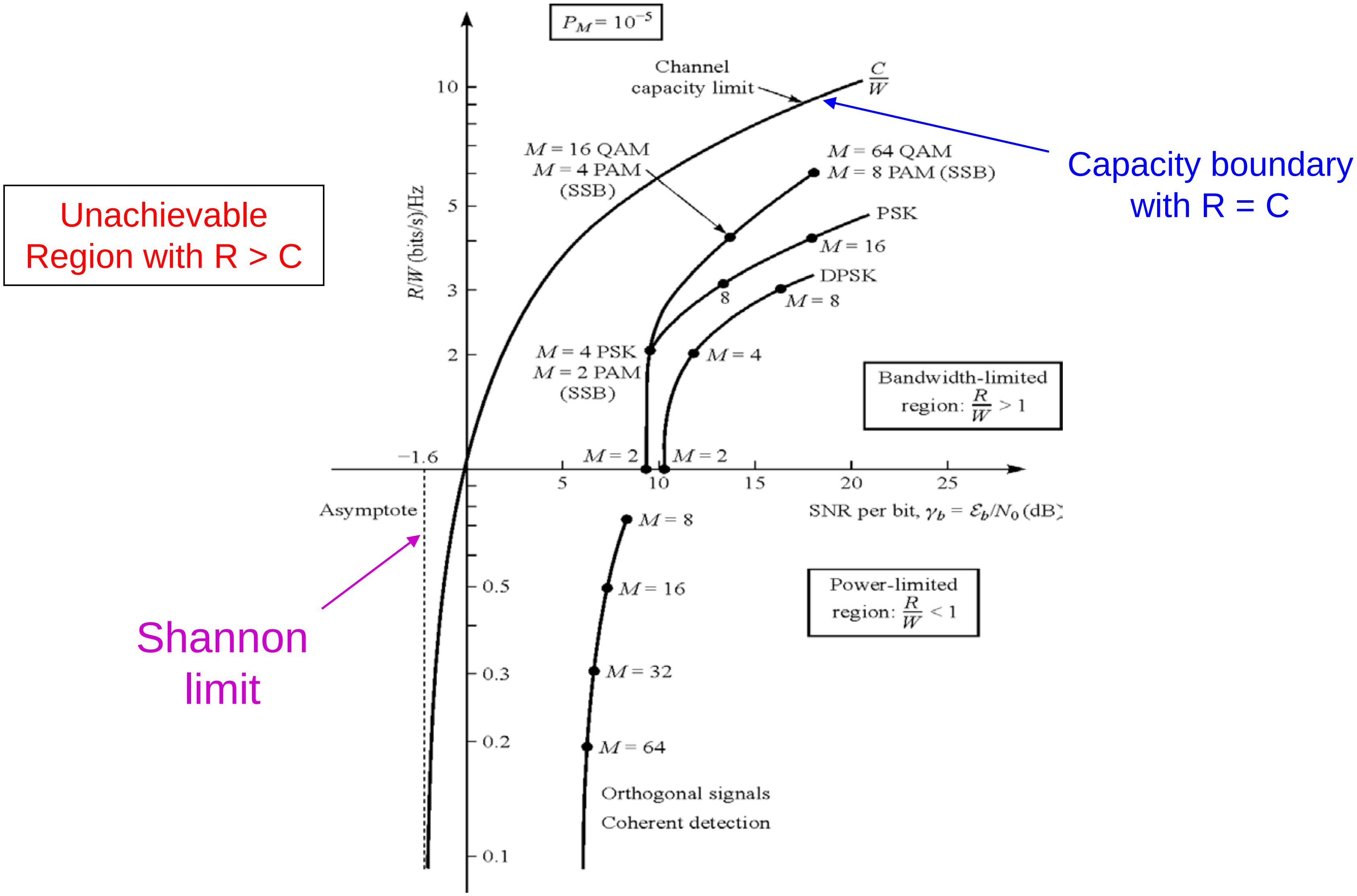
- To see the ultimate power-bandwidth tradeoff, we need to use Shannon's channel capacity theorem:
  - Channel Capacity is the theoretical upper bound for the maximum rate at which information could be transmitted without error *(Shannon 1948)*
  - For a bandlimited channel corrupted by AWGN, the maximum rate achievable is given by

$$R \le C = B\log_2(1+SNR) = B\log_2\left(1+\frac{P_s}{N_0 B}\right)$$

- Note that $$\frac{E_b}{N_0} = \frac{P_s T}{N_0} = \frac{P_s}{RN_0} = \frac{P_s B}{RN_0 B} = SNR\frac{B}{R}$$

- Thus $$\frac{E_b}{N_0} = \frac{B}{R}\left(2^{R/B}-1\right)$$

Power-Bandwidth Tradeoff



Picture from: Proakis J G, Salehi M. Fundamentals of communication systems[M]. Pearson Education India, 2007.

Notes on the Fundamental Tradeoff

- In the limits as R/B goes to 0, we get

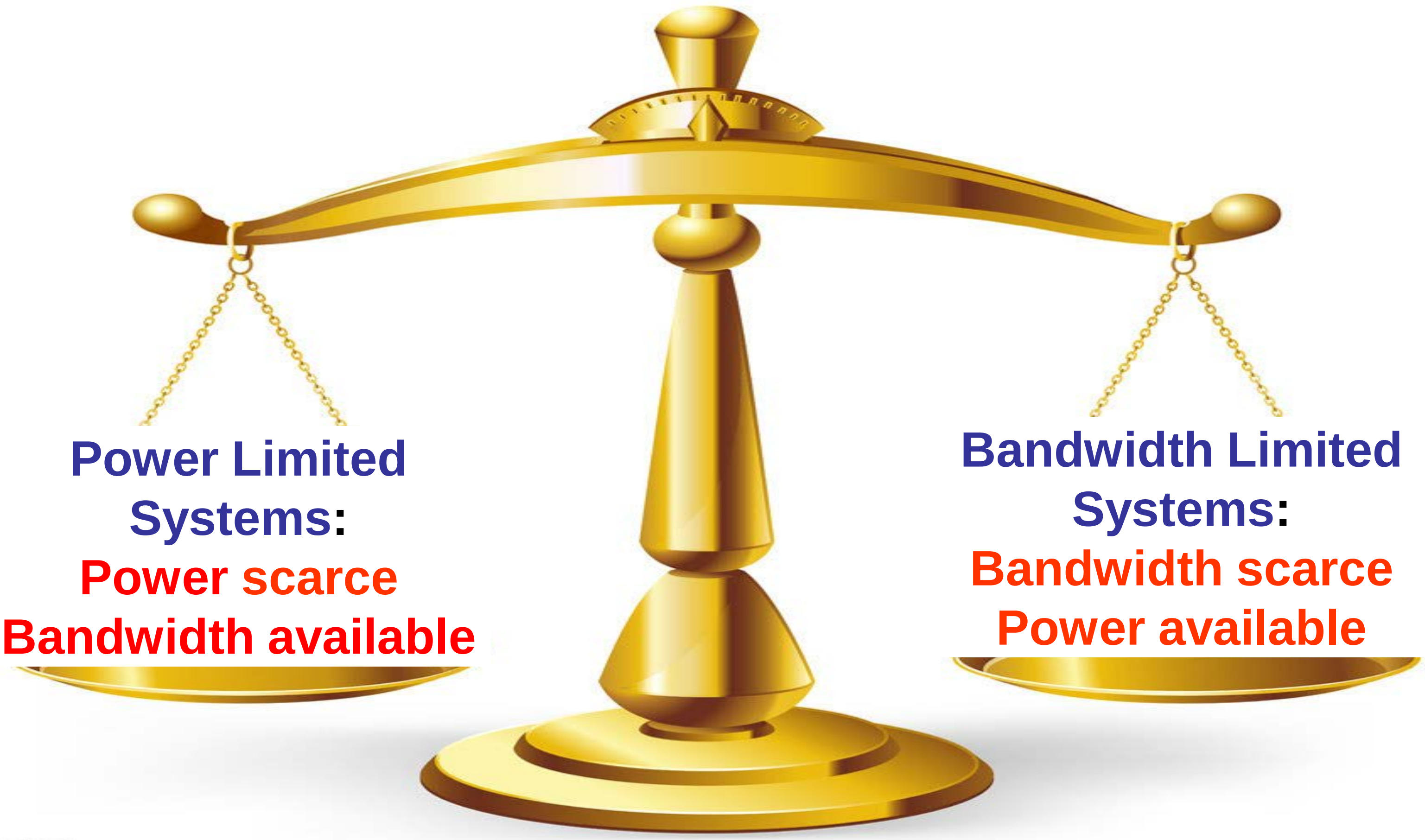
$$\frac{E_b}{N_0} = \ln 2 = 0.693 = -1.59dB$$

  - This value is called the Shannon Limit
  - Received E_b/N_0 must be **>-1.6dB** to ensure reliable communications

- BPSK and QPSK require the same E_b/N_0 of 9.6 dB to achieve $P_e=10^{-5}$. However, QPSK has a better bandwidth efficiency
- MQAM is superior to MPSK
- MPSK/MQAM increases bandwidth efficiency at the cost of lower energy efficiency
- MFSK trades energy efficiency at reduced bandwidth efficiency.

System Design Tradeoff

Which Modulation to Use ?



Power Limited Systems:
Power scarce
Bandwidth available

Bandwidth Limited Systems:
Bandwidth scarce
Power available

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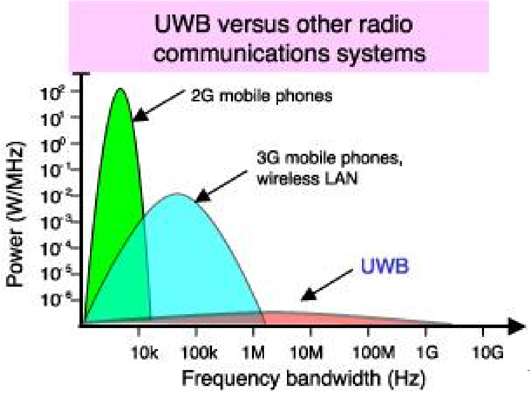

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Practical Applications

- BPSK:
  - WLAN IEEE802.11b (1 Mbps)
- QPSK:
  - WLAN IEEE802.11b (2 Mbps, 5.5 Mbps, 11 Mbps)
  - 3G WDMA
  - DVB-T (with OFDM)
- QAM
  - Telephone modem (16QAM)
  - Downstream of Cable modem (64QAM, 256QAM)
  - WLAN IEEE802.11a/g (16QAM for 24Mbps, 36Mbps; 64QAM for 38Mbps and 54 Mbps)
  - LTE Cellular Systems
- FSK:
  - Cordless telephone
  - Paging system